Semi-toric integrable systems and moment polytopes

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Systèmes intégrables semi-toriques et polytopes moment

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Résumé en français

Cette thèse s'inscrit dans le — vaste — domaine de la géométrie symplectique, domaine qui puise ses racines dans les travaux de mécanique céleste de Lagrange à la fin du XVIII^e siècle (voir l'article [AIZ94] de Michèle Audin et Patrick Iglesias-Zemmour pour un aperçu historique). La géométrie symplectique est intimement liée à la physique : les mathématiciens l'ont développée en cherchant le cadre mathématique le plus naturel pour décrire la mécanique newtonienne des forces conservatives. De fait, tous les formalismes de la mécanique classique se sont unifiés dans le language de la géométrie symplectique, qui est devenue une branche à part entière de la géométrie différentielle dans la deuxième partie du XX^e siècle.

Notre travail porte sur une classe d'objets symplectiques appelés systèmes intégrables semi-toriques, eux-mêmes modelés sur certains exemples de systèmes mécaniques, aussi accessibles et universels que le pendule sphérique (une masse ponctuelle astreinte à se déplacer sur une sphère, seulement soumise à la pesanteur), ou encore différents modèles de toupies (toupie de Lagrange et de Kowalewskaya).

Le but de cette thèse est de faire progresser un programme de classification des systèmes intégrables à l'aide d'objets de nature combinatoire appelés polytopes moment. Cette classification a d'abord été achevée pour le cas des systèmes dits toriques dans les années 80, et étendue au cas plus général des systèmes intégrables dits semi-toriques uniquement en dimension $2n = 4$ dans les années 2000 par San Vũ Ngọc et Álvaro Pelayo.

Les motivations de ce programme sont autant internes à la géométrie symplectique, qu'externes. L'article [PVN11b] décrit ce programme et ses multiples applications, mais nous n'en n'évoquerons que deux.

Tout d'abord, on peut appliquer à ces résultats la théorie de l'analyse semi-classique, et ainsi rendre compte de certains phénomènes de spectroscopie moléculaire ([CDG+04]).

Une autre application de ce programme est la symétrie miroir dans le cadre de la théorie des cordes. En lien avec la formulation dite homologique de cette conjecture reliant les différentes théories des cordes, il existe une conjecture, dite de Strominger-Yau-Zaslov. Celle-ci postule la correspondance entre la catégorie dite de Fukaya d'une variété symplectique et une autre catégorie de sa variété miroir. La catégorie de Fukaya est fortement liée aux systèmes dits presque-toriques (voir l'article de Konsevitch et Soibelman [KS06]). Or les systèmes semi-toriques sont l'exemple le plus simple de systèmes presque-toriques. Une classification de cette classe d'objet fournirait donc des exemples bien compris sur lesquels exercer la conjecture.

De la mécanique à la géométrie symplectique

Les principes de la mécanique newtonienne

Isaac Newton a le premier réussi à formuler la mécanique classique sous forme de lois reliant la position et la vitesse d'un corps à l'ensemble des forces s'appliquant sur ce corps. Aujourd'hui, dans l'enseignement secondaire, les principes de la mécanique newtonienne sont enseignés comme suit :

– **Principe d'inertie :**
  il existe des référentiels dits Galiléens, dans lesquels tout corps soumis à des forces de résultante nulle est soit immobile, soit en mouvement rectiligne uniforme.
– **Principe fondamental de la dynamique :**
  dans un référentiel Galiléen, pour un corps de masse constante m soumis à différentes forces $\vec{F}_i$, $i = 1, \ldots, N$, on a :

$$m\vec{a} = \sum_{i=1}^{N} \vec{F}_i, \text{ où } \vec{a} \text{ désigne } \textit{l'accélération.}$$

– **Principe des actions réciproques :** dans un référentiel Galiléen, tout corps A exerçant sur un corps B une force $\vec{F}_{A/B}$ subit une force $\vec{F}_{B/A}$, du corps B de sens égal à $\vec{F}_{A/B}$ mais de direction opposée :

$$\vec{F}_{A/B} = -\vec{F}_{B/A}$$

Cette formulation permet de traiter beaucoup de problèmes de mécanique courante, mais cette façon d'exprimer les lois physiques de la mécanique classique présente certaines contraintes dont on souhaiterait pouvoir s'affranchir.

Ainsi, Lagrange, dans son mémoire sur les corps célestes [Lag77], utilisait pour décrire le mouvement d'un astre soumis à une force centrale ce qu'on appelle en astronomie les éléments d'une orbite (cinq grandeurs déterminant la conique sur laquelle évolue l'astre dans l'espace, plus une pour situer l'astre sur sa trajectoire à un instant donné). Cependant, il faisait déjà remarquer que ce choix de coordonnées était a priori aussi légitime pour décrire le mouvement des planètes que celui des trois coordonnées de position et de vitesse. La différence tient en la *simplicité* de la description du mouvement des astres dans les premières coordonnées, qui en fait un choix adapté au problème.

La géométrie symplectique est justement la formulation *intrinsèque* des lois la mécanique newtonienne, en dehors de tout choix de coordonnées. Beaucoup de résultats, tels que le célèbre théorème des coordonnées action-angles (voir Chapitre 1, Théorème 1.2.14) portent sur l'existence d'un choix de coordonnées simplifiant considérablement l'expression d'un problème, et donc sa résolution. La formulation intrinsèque du problème signifie que l'on va utiliser les outils de la géométrie dans les variétés différentielles.

Mécanique lagrangienne

La géométrie symplectique émerge naturellement à partir de la reformulation hamiltonienne de la mécanique newtonienne. Une étape intermédiaire est la mécanique lagrangienne, qui exprime le principe fondamental de la dynamique sous la forme d'un principe de moindre action comme le principe de Fermat.

Si l'espace des positions généralisées $(q_i)_{i=1,\ldots,n}$ est une variété N on se place dans l'espace des configuration TN, c'est-à-dire l'espace des positions et vitesses généralisées $(q_i,\dot{q}_i)_{i=1,\ldots,n}$. On définit le Lagrangien $L(q,\dot{q},t)$ en l'absence de champ magnétique comme l'énergie cinétique moins l'énergie potentielle. L'action associée à un chemin $\eta=(q(t),\dot{q}(t))_{t\in[t_1,t_2]}$ est alors

$$\mathcal{S}[\eta]=\int_{t_1}^{t_2}L(q(t),\dot{q}(t),t)dt.$$

La mécanique lagrangienne explique qu'étant donné une condition initiale $(q(t_1),\dot{q}(t_1))$ et finale $(q(t_2),\dot{q}(t_2))$ dans l'espace des configuration, le chemin physiquement réalisé est celui qui minimise l'action $\mathcal{S}$. Une condition nécessaire serait alors que la "dérivée" de $\mathcal{S}$ par rapport à un chemin s'annule pour le chemin minimisant l'action. Lagrange et Euler ont inventé le calcul des variations pour donner un sens à cette notion de dérivée, et en ont tiré les équations dites d'Euler-Lagrange

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right]-\frac{\partial L}{\partial q_i}=0.$$

Mecanique hamiltonienne

On peut remplacer les vitesses par des quantités duales appelées quantités de mouvement $p_i=\frac{\partial L}{\partial \dot{q}_i}$. Cela revient à se placer dans le cotangent T^*N appelé espace des phases. La quantité duale du Lagrangien L est alors appelée le Hamiltonien du système : c'est la transformée de Legendre du Lagrangien

$$H=\sum_{i=1}^{n}\dot{q}_i p_i-L$$

où les vitesses sont supposées écrites en fonction des p_i. On montre alors facilement grâce aux équation d'Euler-Lagrange que le Hamiltonien vérifie les équations dites de Hamilton

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H}{\partial p_i} \ , \ i = 1, \ldots, n \\ \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial q_i} \ , \ i = 1, \ldots, n \end{cases}$$

C'est une équation différentielle ordinaire d'ordre $2n$.

Apparition de la géométrie symplectique

Un théorème de Poincaré indique que le flot de cette équation préserve la somme des aires orientées de la projection sur chacun des plans (x_i, ξ_i). C'est-à-dire, qu'on a conservation de la quantité $\omega_0 = \sum_{i=1}^n dp_i \wedge dq_i$ par le système.

La forme ω_0 est bilinéaire, antisymétrique, non-dégénérée et fermée (elle est même exacte dans ce cas-ci). Cette forme est le point de départ pour définir la géométrie symplectique. Une variété symplectique est une variété différentielle (connexe) de dimension paire M^{2n} munie d'une 2-forme non-dégénérée fermée appelée forme symplectique. Un système hamiltonien est alors une fonction $H \in \mathcal{C}^\infty(M^{2n} \to \mathbb{R})$. On définit le champ de vecteurs hamiltonien X_H associé à H par

$$\imath_{X_H} \omega_0 = \omega_0(X_H, .) = -dH$$

Ce champ de vecteurs, appelé aussi gradient symplectique, est bien défini car ω est non-dégénérée. On a alors immédiatement que le flot du champ de vecteurs X_H est un symplectomorphisme : il préserve ω_0. Le théorème de Poincaré est donc axiomatisé dans la géométrie symplectique. Par la suite on notera plutôt la forme symplectique ω.

Présentation du problème

Systèmes intégrables semi-toriques

On peut introduire une opération sur $\mathcal{C}^\infty(M^{2n} \to \mathbb{R})$ appelée crochet de Poisson. Pour $f, g \in \mathcal{C}^\infty(M^{2n} \to \mathbb{R})$, le crochet de Poisson $\{f, g\}$ est défini comme

$$\{f, g\} = \omega(X_f, X_g) = -df(X_g) = dg(X_f)$$

On appelle intégrale première d'un système hamiltonien H toute fonction f telle que $\{H, f\} = 0$. Une intégrale première de H est donc une fonction constante le long des lignes de champ de X_H. On voit immédiatement que H est une intégrale première de H. On définit alors un système hamiltonien *sous-intégrable* comme la donnée de k intégrales premières f_i dites "en involution" : $\{f_i, f_j\} = 0$, et pour éviter les cas triviaux, on demande que les f_i soient indépendantes pour presque tout $p \in M$. On appelle la fonction $F = (f_1, \ldots, f_k) : M^{2n} \to \mathbb{R}^k$ l'application moment. On pourra déjà remarquer que l'indépendance presque-partout fait que $k \leqslant n$.

Lorsque l'appplication moment du système hamiltonien a n composantes indépendantes presque partout, on dit que le système est **intégrable**. Les surfaces de niveau $F^{-1}(c)$ régulières sont alors des sous-variétés dites lagrangiennes : elles sont de dimension n et la forme symplectique restreinte à ces sous-variétés est identiquement nulle. On note le feuilletage lagrangien associé $\mathcal{F}$.

Un théorème de Noether affirme qu'à chaque intégrale première du système correspond une symétrie intrinsèque de celui-ci (dans sa forme originelle, le théorème de Noether était écrit dans le cadre de la mécanique lagrangienne, mais depuis il est devenu courant d'associer ce théorème à la mécanique hamiltonienne). Un système intégrable est donc un système qui possède un groupe de symétrie suffisament grand pour en permettre une description particulièrement simple. Par exemple, le théorème des coordonnées action-angles s'applique justement pour des systèmes intégrables.

Les sytèmes intégrables sont d'une certaine manière les plus simples des systèmes hamiltoniens, puisqu'ils possèdent autant de symétries que de degrés de liberté. Cependant ils ne recouvrent pas l'ensemble des systèmes mécaniques. Un système de mécanique hamiltonienne peut avoir plus ou moins d'intégrales premières qu'il n'a de degrés de liberté. Pour un système avec "trop" d'intégrales premières, il existe une notion de système dits "super-intégrables". Nous les mentionnons brièvement dans le Chapitre 4. Cependant, la grande majorité des systèmes hamiltoniens ont plutôt "trop peu" d'intégrales premières que "trop". Si l'on prend comme modèle du système solaire le Soleil et l'ensemble des planètes qui gravitent autour, celui-ci n'est pas un système intégrable – entre autre – pour cette raison. Pourtant, beaucoup d'exemples importants de systèmes physiques s'avèrent être des systèmes intégrables. En outre, ils sont un excellent point de départ pour l'étude de systèmes plus complexes mais que l'on peut approcher par des systèmes intégrables. L'étude des systèmes intégrables intéresse donc l'ensemble de la physique.

L'universalité des lois de la mécanique appelle une question : qu'est-ce qui distingue *fondamentalement* un système mécanique d'un autre ? En effet, deux systèmes très différents peuvent présenter des dynamiques similaires : même nombre de points fixes, stables ou instables, mêmes orbites périodiques etc. Aussi, si on s'entend sur une notion d'**équivalence** entre deux systèmes, l'un des objectifs de la géométrie symplectique serait alors de fournir la classification la plus simple possible des systèmes *à équivalence près*, en commençant tout naturellement par les systèmes intégrables.

Une telle classification des systèmes complètement intégrables représente un programme de recherche à très long terme, mais heureusement, d'importantes avancées ont déjà été faites pour des sous-ensembles de systèmes intégrables. Un premier sous-ensemble est celui des systèmes intégrables dit toriques, où le groupe de symétrie donné par le théorème de Noether est un tore $\mathbb{T}^n$. Cela revient à demander que chaque intégrale première ait un flot 2π-périodique.

Pour ces systèmes, Atiyah d'une part ([Ati82]), Guillemin et Sternberg d'autre part ([GS82],[GS84]), ont démontré que l'image de l'application moment était un polytope convexe à faces rationnelles. Une étape cruciale dans la preuve était la démonstration de la connexité des fibres de l'application moment.

Peu de temps après, Delzant a démontré dans les deux articles [Del88] et [Del90] que, sous certaines conditions peu restrictives, ce polytope permettait de classifier complètement les systèmes intégrables toriques de façon très fine. En effet deux systèmes toriques sont dits équivalents s'il existe un symplectomorphisme préservant l'application moment. En outre, la preuve est constructive : si on se donne un polytope Δ satisfaisant certaines conditions (de tels polytopes sont aujourd'hui appelés polytopes de Delzant), alors on peut construire un système intégrable, c'est-à-dire une variété symplectique M^{2n} et une application moment $F : M^{2n} \to \mathbb{R}^n$, telle que $F(M) = \Delta$.

Il est plus difficile de définir la notion de semi-toricité ou même d'en donner l'intuition. Sa définition rigoureuse est l'objet d'une large partie du Chapitre 1. On peut néanmoins donner l'heuristique suivante : pour un système intégrable, les points critiques de F peuvent présenter des composantes dites elliptiques, qui sont stables dynamiquement, hyperboliques, qui sont instables, et foyer-foyer. Les composantes foyer-foyer sont spécifiques aux systèmes dynamiques définis sur des variétés symplectiques. Elles ont des variétés stables et instables, comme les composantes hyperboliques mais sont de natures différentes de ces dernières : elles possèdent une symétrie circulaire qu'il est impossible de séparer de la dynamique "instable".

Par exemple, dans le pendule sphérique sans frottements, si la masse se situe au pôle Nord de la sphère avec une vitesse nulle, on a clairement un point fixe instable, mais symétrique par rotation autour de l'axe Nord-Sud : c'est notre point fixe foyer-foyer.

Les points critiques d'un système intégrable torique ont uniquement des composantes elliptiques. Les composantes foyer-foyer sont plus faciles à analyser que les composantes hyperboliques, c'est pourquoi l'introduction de points critiques avec des composantes foyer-foyer est une extension très naturelle du cadre torique.

Pour les systèmes intégrables semi-toriques, on s'autorise donc des points critiques de F avec des composantes foyer-foyer, mais on restreint les possibilités d'apparition de celles-ci. D'abord, on demande que l'équilibre instable n'intervienne que sur une seule composante (par convention, la première). On demande en outre que les autres composantes $f_2, \ldots, f_n$ de l'application moment fournissent un tore $\mathbb{T}^{n-1}$ comme groupe de symétrie globale pour le système.

La question naturelle est alors de chercher à généraliser les théorèmes d'Atiyah – Guillemin & Sternberg et de Delzant aux systèmes intégrables semi-toriques. Ceci a été la motivation principale de cette thèse, et plusieurs résultats intermédiares importants ont été obtenus dans cette direction.

Les résultats existants

La principale difficulté pour étendre le résultat d'Atiyah – Guillemin & Sternberg vient de ce que la présence de singularités foyer-foyer induit un phénomène dit de monodromie dans le feuilletage lagrangien donné par les ensembles de niveau de l'application moment. La première conséquence de cette monodromie est que l'image de l'application moment n'est plus un polytope, elle n'est même plus convexe a priori. Cependant, il existe par hypothèse un sous-tore de dimension $n-1$ comme groupe de symétrie globale. La question se pose donc de savoir ce qui a survécu du polytope avec la perte d'une action de S^1, et comment le cas échéant récupérer celui-ci à partir de l'image. D'autre part, on peut se douter que, à l'instar du cas torique, la connexité des fibres va jouer un rôle crucial dans les résultats. Il faut donc examiner si l'on peut redémontrer celle-ci dans le cas semi-torique.

San Vũ Ngọc a résolu le problème en dimension $2n = 4$ dans l'article [VN07]. Il a démontré la connexité des fibres, et il a donné une méthode où l'on découpe et redresse l'image pour récupérer un polytope de Delzant. Seulement, le découpage n'étant pas unique, on se retrouve désormais avec une famille finie de polytopes ayant une structure de groupe isomorphe à $(\mathbb{Z}/(2\mathbb{Z}))^{m_f}$ où m_f désigne le nombre de singularités foyer-foyer.

Ensuite, en ce qui concerne la classification, il faut d'abord définir une équivalence spécifique aux systèmes intégrables semi-toriques. Deux systèmes intégrables semi-toriques sont dits ST-équivalents s'il existe un symplectomorphisme ramenant le feuilletage de l'un sur l'autre, et qui préserve exactement les $n-1$ dernières composantes de l'application moment.

Dans l'article [PVN09], San Vũ Ngọc et Álvaro Pelayo ont réussi à produire pour un système intégrable semi-torique de dimension $2n = 4$ une liste d'invariants comprenant la famille de polytopes ci-dessus. Elle contient aussi des invariants définis par San Vũ Ngọc dans [VN03], et qui décrivent comment le feuilletage se singularise au voisinage de chaque singularité foyer-foyer. Plus précisément, les deux auteurs montrent que deux systèmes intégrables semi-toriques sont ST-équivalents au sens ci-dessus si et seulement si tous les invariants de la liste sont identiques. Dans l'article [PVN11a], les mêmes auteurs se fixent la même liste d'invariants $\mathcal{L}$, et construisent un système intégrable semi-torique dont la liste d'invariants associée est $\mathcal{L}$. On a donc un résultat de classification à la Delzant pour les systèmes intégrables semi-toriques en dimension $2n = 4$.

L'objectif de cette thèse est de proposer une extension des résultats énoncés ci-dessus en dimension quelconque. Même si la classification complète reste encore hors de notre portée, nous disposons de plusieurs résultats intermédiaires (voir Chapitre 3). Nous avons en outre ouvert de nouvelles perspectives au programme de recherche initial (voir Chapitre 4 et 5 par exemple)

D'autres résultats dans lesquels s'inscrivent cette thèse sont à mentionner ici. Tout d'abord, les travaux de Nguyen Tien Zung sur la classification topologique et symplectique des fibrations lagrangiennes singulières. Dans un premier article ([Zun96]) Nguyen Tien Zung démontre un théorème de coordonées action-angles en présence de singularités non-dégénérés (voir Chapitre 3, Theorem 3.1.11), puis il utilise ce résultat dans un autre article ([Zun03]) pour produire une classification topologique et symplectique des fibrations lagrangiennes singulières non-dégénérés (avec d'autres hypothèses de généricité).

Zung repart en fait des travaux de Duistermaat ([Dui80a]), Dazord et Delzant ([DD87]) l'obstruction à l'existence de variables action-angles régulières globales pour un système intégrable. Il définit plusieurs objets, dont la monodromie (pour être exact le faisceau de monodromie) du système, et des classes caractéristiques appelées classes de Chern-Duistermaat et de Chern-Duistermaat lagrangienne. Il montre alors que deux fibrations singulières sont "symplectiquement équivalentes" si et seulement si les trois objet coïncident. En effet, ces trois objets sont les obstructions à ce que le feuilletage $\mathcal{F}$ soit un fibré en tores lagrangiens trivial.

Zung indique lui-même dans [Zun03] qu'il est plus intéressé par le feuilletage $\mathcal{F}$ que par l'application moment F dont ce dernier est issu. Pour appliquer les résultats de Zung à une classification des systèmes intégrables, il faut supposer la connexité des fibres de F. Or pour nous, cette connexité est un des problèmes à résoudre. Une autre manière de dire les choses est que l'objet de départ pour nous est l'application moment, et pas le feuilletage lagrangien associé. En outre, appliquée au cas semi-torique, cette classification ne tire pas parti des spécificités de ces systèmes, comme les $n-1$ intégrales premières qui sont 2π-périodiques. Comparée à la ST-équivalence, la classification de Zung est donc plus grossière.

Enfin, une autre approche est celle adoptée par Yael Karshon et Sue Tolman. Dans une série d'articles ([Kar99],[KT01],[KT11]) les auteures classifient les variétés symplectiques M^{2n} munies d'une action d'un tore $\mathbb{T}^{n-1}$, à symplectomorphisme équivariant près. La classification est dans l'esprit de celle que nous recherchons, avec une liste finie d'objets de nature combinatoire. Certains de ces invariants tels que la mesure de Duistermaat-Heckmann sont aussi présents dans les travaux de San Vũ Ngọc et Álvaro Pelayo.

Cette classification englobe tous les systèmes sous-intégrables à $n-1$ composantes, et traite donc le cas du sous-système $\check{F} = (f_2, \dots, f_n)$, mais sans tirer parti de ce que le système est en réalité intégrable. Là encore, la classification est trop large. En outre, on n'a toujours pas de résultat de connexité des fibres de F.

Il serait intéressant de voir les systèmes intégrables semi-toriques comme une fonction supplémentaire f_1 en involution avec $\check{F}$, mais le fait que les composantes foyer-foyer entremêlent nécessairement f_1 et $\check{F}$ nous a convaincu de traiter le système sans séparer les composantes. Il n'en reste pas moins que dans cette thèse, nous nous sommes appuyés sur plusieurs résultats démontrés d'abord pour des systèmes sous-intégrables, notamment dans le Chapitre 4.

Un dernier résultat à mentionner est la classification donnée par Symington et Leung dans [Sym01] et [LS10] des variétés symplectiques de dimension 4 sur lesquelles on peut définir un feuilletage comprenant uniquement des singularités foyer-foyer ou elliptiques. Ces travaux nous ont été très utiles pour la description détaillée de la structure affine singulière portée par l'espace de base du feuilletage $\mathcal{F}$ (voir Chapitre 5).

Les résultats obtenus dans cette thèse

Dans cette thèse, nous avons obtenu plusieurs résultats qui devraient nous permettre prochainement de récupérer une famille de polytopes à partir de l'image de l'application moment $F(M)$ comme dans l'article [VN07]. D'autres résultats obtenus durant la thèse devraient s'avérer utiles pour la classification des systèmes intégrables semi-toriques.

Tout d'abord, nous fournissons au Chapitre 2 une preuve complète du théorème de forme normale d'Eliasson dans le cas d'un point fixe foyer-foyer.

Soit m un point fixe foyer-foyer d'un système intégrable $F = (f_1, f_2) : M^4 \to \mathbb{R}^2$, c'est-à-dire tel que :

- $df_1(m) = df_2(m) = 0$;
- Il existe un symplectomorphisme linéaire

$$(T_m M, (\mathcal{H}_m(f_1), \mathcal{H}_m(f_2))) \text{ et } \mathcal{Q}_f = (\mathbb{R}^4, aq_1 + bq_2, cq_1 + dq_2),$$

avec

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}),\ q_1 = x_1\xi_1 + x_2\xi_2 \text{ et } q_2 = x_1\xi_2 - x_2\xi_1.$$

On démontre alors le théorème suivant (Chapitre 2, Théorème 2.1.2) :

Théorème 1 (Forme normale d'Eliasson, cas foyer-foyer)**.** *Soit m un point fixe foyer-foyer d'un système intégrable $F = (f_1, f_2) : M^4 \to \mathbb{R}^2$.*

Alors dans un voisinage de m il existe un symplectomorphisme local Ψ : $(\mathbb{R}^4, \omega_0) \to (M^4, \omega)$ et un difféomorphisme local G d'un voisinage de $F(m)$ tel que

$$F \circ \Psi = G(q_1, q_2).$$

Ce théorème, et sa généralisation au cas de points critiques foyer-foyer-elliptiques et foyer-foyer-transverses (en abrégé $FF - E$ et $FF - X$), donne une sorte de modèle local de l'image de l'application moment au voisinage de valeurs critiques $FF - E$ et $FF - X$. C'est en utilisant ces modèles locaux dans le Chapitre 3 que nous avons produit un premier résultat, inédit à notre connaissance. On définit de la manière suivante les possibles valeurs critiques de l'image de l'application moment :

Définition 1. *Dans l'espace affine $\mathbb{R}^3$, une courbe gauche de valeurs $FF - X$ est appelée un* chemin nodal*.*

On a ensuite la description suivante de l'image de l'application moment (Théorème 3.2.10) :

Théorème 2. *Soit F un système intégrable semi-torique sur une variété symplectique compacte M^6. L'image de l'application moment se situe dans l'espace affine $\mathbb{R}^3$. On a alors :*

1. *Les valeurs foyer-foyer-elliptique ($FF - E$) forment un ensemble fini de points.*
2. *L'ensemble des valeurs foyer-foyer-transverses ($FF - X$) est une union finie de chemins nodaux :*

$$\mathrm{CrV}_{FF-X}(F) = \bigcup_{i=1}^{m_\mathrm{f}} \gamma_i \ , \ m_\mathrm{f} \in \mathbb{N}.$$

3. *Chaque chemin nodal γ_i est contenu dans un plan affine à coordonnées entières $\mathcal{P}(\gamma_i)$. Dans le plan $\mathcal{P}(\gamma_i) = A_i + \mathbb{R} \cdot e_1 + \mathbb{R} \cdot e_2$, le chemin γ_i peut s'exprimer comme le graphe d'une fonction lisse d'un intervalle $]0, 1[$:*

$$\gamma_i = \{A_i + t \cdot \vec{e_1} + h_i(t) \cdot \vec{e_2} \ , \ t \in]0, 1[\ \}$$

Les limites en 0 et en 1 du chemin nodal sont des valeurs $FF - E$.

Ce théorème est de très bon augure pour la suite. En effet, en dimension $2n = 4$ on découpait l'image par des demi-droites au dessus ou en dessous des points foyer-foyer. La conjecture en dimension supérieure est que l'on va devoir découper l'image de l'application moment dans ces plans, selon l'épigraphe ou l'hypographe du chemin nodal.

Nous n'avons donné dans cette thèse une démonstration de ce résultat que pour le cas $2n = 6$ mais les résultats intermédiaires sur les modèles locaux sont eux démontrés en dimension quelconque. La raison en est qu'entre temps nous avons pris conscience qu'il était possible de fournir une version encore plus générale desdits résultats, en s'appuyant sur le théorème d'Atiyah – Guillemin & Sternberg.

En effet, ce théorème s'applique également pour des systèmes sous-intégrables. Or nous avons démontré l'existence d'une stratification de M par des sous-variétés symplectiques, sur lesquelles on peut définir des systèmes semi-toriques "extraits" de F. Le résultat obtenu (voir Chapitre 4, Section 4.1.1) est une première extension de Atiyah – Guillemin & Sternberg au cas semi-torique. S'il ne suffit pas à redonner immédiatement le polytope moment, il nous a permis avec des arguments de théorie de Morse de démontrer le théorème suivant (voir Chapitre 4, Théorème 4.2.4), sans doute le résultat principal de cette thèse.

Théorème 3. *Soit un système intégrable semi-torique $F : M^{2n} \to \mathbb{R}^n$. Alors les fibres $F^{-1}(c)$, $c \in F(M)$, sont connexes.*

Le Chapitre 5 est un peu à part dans cette thèse, puisqu'il porte sur l'espace de base $\mathcal{B}$ du feuilletage lagrangien singulier, dans le cas général d'un système intégrable dit fortement non-dégénéré, et pas sur l'image de l'application moment. Cet espace bien que n'étant pas une variété régulière, possède une structure très riche. En effet il possède une structure affine entière stratifiée.

Nous avons décrit en détail cette structure à l'aide du langage des faisceaux, et nous avons nommé de tels espaces des *stradispaces $\mathbb{Z}$-affines*. Nous avons ensuite démontré que l'espace de base d'un système intégrable semi-torique était un *stradispace $\mathbb{Z}$-affine*. Nous avons alors repris la notion de "convexité intrinsèque" définie par Zung dans l'article [Zun06a] pour ces espaces. Il s'agit de définir une notion de convexité relative à une structure affine, même singulière comme dans notre cas. Nous avons montré alors que pour un système intégrable semi-torique, l'espace de base $\mathcal{B}$ était localement intrinsèquement convexe, ce que nous voyons comme un résultat préliminaire à la convexité intrinsèque *globale* de l'espace de base du feuilletage.

Ce résultat nous intéresse à plus d'un titre, notamment au regard des théorèmes prouvés dans [Zun06a]. En effet, un théorème y énonce qu'étant donné un stradispace affine $(X, \mathcal{A}, \mathcal{S})$ intrinsèquement convexe, et n applications affines $u = (u_1, \ldots, u_n)$ de $(X, \mathcal{A}, \mathcal{S})$ dans $\mathbb{R}^n$ muni de sa structure affine usuelle, alors $u(X)$ est convexe au sens usuel.

Appliqué au cas torique, il permet de redémontrer facilement le théorème d'Atiyah – Guillemin & Sternberg. Dans le cas semi-torique, on dispose uniquement des $n - 1$ applications affines données naturellement par le problème. Il manque justement l'étape du "découpage" pour obtenir une n-ième fonction. Cependant à partir de la convexité intrinsèque de $\mathcal{B}$ on peut également démontrer facilement que les fibres de F sont connexes.

Tous ces résultats font donc de l'existence d'une famille de polytopes obtenue à partir de l'image $F(M)$ une conjecture qui semble raisonnable. Dans le Chapitre 3 nous avons également étendu au cas $FF - X^{n-2}$ un théorème fournissant des coordonnées action-angles explicites au voisinage d'une singularité FF. On détermine un sous-système de coordonnées action-angles qui survivent sur la fibre singulière, et une asymptotique explicite des coordonnées qui divergent lorsque les actions s'approchent du chemin nodal. Ce théorème permet entre autre de calculer la monodromie du feuilletage $\mathcal{F}$ autour d'un chemin nodal.

Chapter 1

Semi-toric integrable systems

As a preliminary remark, we have to mention a terminology problem that is not solved. The term "semi-toric integrable systems" designate almost-toric integrable systems of complexity 1, that is, the ones that present a global $\mathbb{T}^{n-1}$-action. The term almost-toric as well as the rest of the terminology was first introduced by Symington in [Sym01], but the word "semi-toric" was first introduced by San Vũ Ngọc in [VN07], where he treated the case $2n = 4$. It suited well the problem, as the system really was "half toric": we had a global $\mathbb{T}^1$-action, when the toric case supposed a $\mathbb{T}^2$-action. The term also had the advantage of simplicity.

However, in higher dimension, even if the term "almost-toric" still fits, there is no more fortunate coincidence with the name "semi-toric". Given that almost-toric integrable systems of a given complexity c represent a class of systems of interest, but given also that the case $c \geqslant 2$ appears to be much more complicated to treat than the case $c = 1$, we propose the following terminology:

A c**-almost-toric system** *designates an almost-toric integrable Hamiltonian system that induces a global Hamiltonian action of a torus of dimension* $n - c$*. The term "semi-toric" is reserved to the simpler case* $c = 1$*.*

In this chapter, we recall all the notions of symplectic geometry which are necessary to a rigorous definition of semi-toric integrable systems, as well as the important results already existing in the literature that we will refer to frequently. A more complete and detailed introduction of the subject can be found in [MS99].

1.1 Symplectic geometry

A symplectic manifold can be understood as the minimal geometrical setting to modelize the equations of Hamiltonian motion described in the Introduction. In this thesis, we will always suppose that M is connected.

Definition 1.1.1. *A symplectic vector space is a (real) vector space endowed with a bilinear form that is skew-symmetric and non-degenerate. A symplectic manifold is a differential manifold M equipped with a closed non-degenerate 2-form ω. A local diffeomorphism $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ is a symplectomorphism if:*

$$\varphi^* \omega_2 = \omega_1$$

Remark 1.1.2. *For every $p \in M$, (T_pM, ω_p) is a symplectic vector space. The 2-form ω depends smoothly of p, and its closedness can be seen as an integrability condition.*

Because of the existence of ω, M is always of even dimension, so we'll usually denote the dimension by $2n$, and call n the (number of) degrees of freedom.

Example 1.1.3. *The following are examples of symplectic manifolds:*

- $(\mathbb{R}^{2n}_{(x_i,y_i)_{i=1,\ldots,n}}, \omega_{\mathbb{R}} = \sum_{i=1}^n dy_i \wedge dx_i)$,
- *Any orientable 2-manifold: the symplectic form is then just the surface form,*
- *If N^d is a differential manifold, then the cotangent bundle $\pi : M = T^*N \to N$ is endowed with a natural 1-form Θ called the tautological 1-form: let $T\pi : TM \to TN$ be the induced tangent map. Let m be a point on M. Since M is the cotangent bundle, we can understand m to be a linear form on the tangent space at $q = \pi(m)$: $m = T_qN \to \mathbb{R}$. We define then $\Theta_m = m \circ T\pi$ as a linear map with values in $\mathbb{R}$, and hence $\Theta \in \Gamma(T^*M)$. We have that $\omega = d\Theta$ is a symplectic form. If $q_1, \ldots, q_n$ are local coordinates and $u_1, \ldots, u_n$ the corresponding cotangent coordinates, we have $\Theta = \sum_{i=1}^d u_i dq_i$ and $\omega = \sum_{i=1}^d du_i \wedge dq_i$,*
- *A complex manifold X: if $(z_1, \ldots, z_k)$ is a local system of coordinates, we can define a symplectic form on X as $\omega_X = \sum_{i=1}^k \Re e(dz_i \wedge d\bar{z}_i)$.*

Note that in the two first examples above, the symplectic form is exact: ω can be written globally as $\omega = d\lambda$ where λ is a 1-form. Such a 1-form is called a Liouville 1-form. It always exists locally because of the closedness of ω, but in general it does not exists globally (e.g.: the sphere S^2 equipped with $\omega = d\theta \wedge d\varphi$ with (θ, φ) the usual spherical coordinates). Actually, on any compact symplectic manifold (M^{2n}, ω), the 2-form ω is not exact (this is an immediate application of Stokes' formula to the volume form $|\omega^n|$).

Definition 1.1.4. *For $H \in \mathcal{C}^\infty(M \to \mathbb{R})$, the* **symplectic gradient** *of H is defined as the vector field $X_H \in \Gamma(TM)$ such that:*

$$\imath_{X_H}\omega = -dH$$

Notice that the vector field X_H is well-defined since ω is non-degenerate.

Conversely, a vector field $Y \in \Gamma(TM)$ is called symplectic *if $\mathcal{L}_Y\omega = 0$. It is called* Hamiltonian *if there exists $H \in \mathcal{C}^\infty(M \to \mathbb{R})$ such that Y is the symplectic gradient of H.*

It is straightforward to see that every Hamiltonian vector field is symplectic. We give below standard examples of Hamiltonian and symplectic vector fields.

Examples 1.1.5.

- *On the standard symplectic 2-sphere $(\mathbb{S}^2, dh \wedge d\theta)$ the vector field $X_h = \dfrac{\partial}{\partial\theta}$ is Hamiltonian. Its Hamiltonian is, up to a constant, given by the height function:*

$$\imath_{X_h}(dh \wedge d\theta) = -dh$$

The motion generated by this Hamiltonian is the rotation around the vertical axis, which preserves the area as well as the height. Note that X_h is well-defined on $\mathbb{S}^2$ even though the variable θ is not defined on the North and South poles.

- *On the symplectic 2-torus $(\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$, the vector fields $X_1 = \dfrac{\partial}{\partial\theta_1}$ and $X_2 = \dfrac{\partial}{\partial\theta_2}$ are symplectic but not Hamiltonian.*

The next theorem gives an important feature of symplectic geometry, which makes it very different from Riemannian geometry.

Theorem 1.1.6 (Darboux)**.** *If (M^{2n}, ω) is a symplectic manifold of dimension $2n$, then for any $p \in M$, there exists a neighborhood $\mathcal{U}_p$ of p and a symplectomorphism $\varphi : (\mathcal{U}_p, \omega) \to (\mathbb{R}^{2n}, \omega_\mathbb{R})$. Such a symplectomorphism is called a Darboux chart, or Darboux coordinates.*

Indeed, this theorem shows that "there is no local theory" in symplectic geometry. In contrast to this, the curvature is a $\mathcal{C}^2$ local invariant of a Riemannian manifold.

Definition 1.1.7. *Let (M^{2n}, ω) be a symplectic manifold. For any linear subspace W of a symplectic vector space (V, ω_V), we define*

$$W^{\perp\omega_V} := \{y \in V \mid \forall x \in W\ ,\ \omega_V(x, y) = 0\}$$

the symplectic orthogonal set of W.

A submanifold N of M is called:

– **isotropic** *if* $\forall p \in N$, $T_pN \subseteq T_pN^{\perp\omega_p}$, *or* $i_N^*\omega = 0$,
– **coisotropic** *if* $\forall p \in N$, $T_pN \supseteq T_pN^{\perp\omega_p}$,
– **Lagrangian** *if* $\forall p \in N$, $T_pN = T_pN^{\perp\omega_p}$*: N is maximally isotropic, minimally coisotropic.*

The last notion will be of particular interest for us, as the common level sets of an integrable system are a foliation by Lagrangian submanifolds.

1.2 Hamiltonian systems in symplectic geometry

From now on and except mention of the contrary, M will always denote a symplectic manifold of dimension $2n$ and with symplectic form ω. A function $H \in \mathcal{C}^\infty(M \to \mathbb{R})$ defines a Hamiltonian system with the following system of differential equations:

$$\text{(Ham)} \begin{cases} \text{Initial condition: } \phi_H^0(m) = m \\ \dfrac{d}{dt}(\phi_H^t(m)) = X_H(\phi_H^t(m)) \end{cases}$$

For a fixed t, the function $m \mapsto \phi_H^t(m)$ is a diffeomorphism. It is called the **flow** of H, as it describes the motion of a point along the integral curves of (Ham) as the time "flows". If we write the equations of a local system in Darboux local coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$, setting

$$\phi_H^t = (x_1(t), \dots, x_n(t), \xi_1(t), \dots, \xi_n(t))$$

for the flow, we have

$$\text{(Ham)} \iff \begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial \xi_i} \ , \ i = 1, \dots, n \\ \dfrac{d\xi_i}{dt} = -\dfrac{\partial H}{\partial x_i} \ , \ i = 1, \dots, n. \end{cases} \tag{1.1}$$

These are exactly Hamilton's equations of motion. With the next tool, we give another way of writing Hamilton's equations

Definition 1.2.1. *For $f, g \in \mathcal{C}^\infty(M \to \mathbb{R})$, we define the Poisson bracket*

$$\{f, g\} = \omega(X_f, X_g) = -df(X_g) = dg(X_f)$$

Equipped with $\{\cdot,\cdot\}$, $(\mathcal{C}^\infty(M \to \mathbb{R}), \{\cdot,\cdot\})$ is a Poisson algebra.

We state the following properties without proof

Proposition 1.2.2.

- *In Darboux coordinates, the Poisson bracket is given by the following formula:*
$$\{f,g\} = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i} \right]$$
- *The Poisson bracket satisfies the Jacobi identity*
$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0. \tag{1.2}$$
- $[X_f, X_g] = X_{\{f,g\}}$
- $0 \to \mathbb{R} \hookrightarrow^{cstt} \mathcal{C}^\infty(M \to \mathbb{R}) \to \chi^1(M)$ *is a Lie algebra morphism.*
- $\{.,f\} : \mathcal{C}^\infty(M \to \mathbb{R}) \to \mathcal{C}^\infty(M \to \mathbb{R})$ *is a derivation. It is actually the Lie derivative with respect to* X_f*.*

The last property allows us to rewrite Hamilton's equations where x and ξ play now a symmetric role. With the same notations used in Equation 1.1, we have

$$(\text{Ham}) \Longleftrightarrow \begin{cases} \dfrac{dx_i}{dt} = \{x_i, H\}\ ,\ i = 1,\dots,n \\ \dfrac{d\xi_i}{dt} = \{\xi_i, H\}\ ,\ i = 1,\dots,n \end{cases} \tag{1.3}$$

Poisson bracket is more than just a commodity of writing equations of motion. Its importance has led to an axiomatic definition of Poisson bracket where we no longer need a symplectic structure to rely on. This is not our concern in this thesis. The Poisson bracket is involved in our problem because of the following definition.

Definition 1.2.3. *A Hamiltonian* $H \in \mathcal{C}^\infty(M \to \mathbb{R})$ *is said to be* **integrable** *(or Liouville integrable, or completely integrable) if there exists* $f_1, \dots, f_n \in \mathcal{C}^\infty(M \to \mathbb{R})$ *such that*
- $\{H, f_i\} = 0\ \forall i = 1,\dots,n$ *(first integrals of* H*),*
- $df_1 \wedge \dots \wedge df_n \neq 0$ *everywhere but on a set of measure zero (linear independance),*
- $\forall\ i,j = 1,\dots,n\ ,\ \{f_i, f_j\} = 0$ *(Poisson commutation).*

$F = (f_1, \dots, f_n)$ *is called the* moment map *of the Hamiltonian.*

The first integrals encode the symmetries of the Hamiltonian: a theorem of Noether states that if H is invariant under the action of a Lie group G, there exists linear combinations of the first integrals such that the action of G on M is generated by the Hamiltonian flows of these combinations. This idea is developed in Section 1.3.

Integrable Hamiltonian systems are a very small subclass of dynamical systems, even of Hamiltonian ones. Yet, integrability plays a central role in the theory, as it is in some sense the least restrictive assumption one can make on a dynamical system to ensure a "solvability" of the system.

Since our interest is in the moment map of integrable systems, we will only use the following definition:

Definition 1.2.4. *A* completely, of Liouville integrable system *is defined as a n-uplet $F = (f_1, \ldots, f_n)$ of functions in $\mathcal{C}^\infty(M \to \mathbb{R})$ such that:*

1. *$rank(dF) = n$ almost everywhere in M^{2n},*
2. *$\forall i, j = 1, \ldots, n$, $\{f_i, f_j\} = 0$.*

The map $F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ is called the moment map *of the system. We denote the (Abelian) subalgebra of $\mathcal{C}^\infty(M \to \mathbb{R})$ generated by the f_i's as: $\mathbf{f} = \langle f_1, \ldots, f_n \rangle$. Since $rank(dF) = n$ almost everywhere in M^{2n}, we say that the* rank *of $\mathbf{f}$ is n. In the more general case where we only have $k \leqslant n$ of such functions, we speak of a* sub-integrable system.

Remark 1.2.5. *It turns out that a number of definitions and assertions concerning (sub-)integrable systems only depend on $\mathbf{f}$ and not on a particular moment map (a basis of $\mathbf{f}$). We shall mention it each time this is the case, to give another point of view on the matter.*

Remark 1.2.6. *If the flows $\phi^t_{f_i}$ are complete, the integrable systems give a Hamiltonian action of $\mathbb{R}^n$ ($\mathbb{R}^k$ in the sub-integrable case).*

We will see in the section related to Hamiltonian actions that there exists a more general notion of moment map, and in some sense, more intrinsic. Yet, we chose this presentation, as we are firstly interested in integrable systems.

There is a rich literature concerning integrable systems, and we cannot provide a comprehensive bibliography. However, we will cite the following books, that should convince the reader that integrable systems are at the crossroad of many paths: apart from the famous *Mathematical Methods of Classical Mechanics* [AWV89], we may cite [AM78], [VN06] for the links with semi-classical analysis, [Aud08] for aspects concerning differential Galois theory, including the famous Non-integrability Theorem of Morales & Ramis, which can be summarized as follow: an integrable Hamiltonian system must have a "simple enough" Differential Galois group. We should also cite [Aud96] and [Aud03], two books in which the author focused on examples rather than raw theory.

1.2.1 Isomorphisms and equivalence relations

There are two objects associated to a Hamiltonian system: the moment map, and the foliation associated to it. Each of them accounts for the analytic and geometric nature of integrable systems. The foliation is defined through the moment map, but we must make a clear distinction between the two of them as we shall see.

Definition 1.2.7. *For a moment map F, we define the following relation $\sim_{\mathcal{F}}$ on M:*

$$x \sim_{\mathcal{F}} y \Longleftrightarrow x \text{ and } y \text{ are in the same connected component of } F^{-1}(F(x))$$

This is an equivalence relation. The set of equivalence classes for this relation is called the base space*: $\mathcal{B} := M/\sim_{\mathcal{F}}$. It is the leaf space of the foliation $\mathcal{F}$.*

The moment map then factors through $\mathcal{B}$ by the following commuting diagram:

$$\begin{tikzcd} M \arrow[r, "\pi_{\mathcal{F}}"'] \arrow[dr, "F"'] & \mathcal{B} \arrow[d, "\tilde{F}"] \\ & F(M) \subset \mathbb{R}^n \end{tikzcd}$$

The map $\pi_{\mathcal{F}}$ is what we refer to as the foliation of the Hamiltonian system, while F is the fibration. Leaves are noted $\Lambda_b = \pi_{\mathcal{F}}^{-1}(b)$. Note that these are in general singular submanifolds due to the presence of critical points of F.

We will also see that, for integrable systems, $\mathcal{B}$ is actually a space with very rich structures on it: it is an *integral affine stradispace*. Chapter 5 is dedicated to their study. We also see that $\hat{F}$ is a local isomorphism with respect to these structures, and thus allows to transport a lot of results from $\mathcal{B}$ to $F(M)$. A lot of work in this field ([PRVN11]), including this thesis (e.g.: Theorem 4.2.4), investigates under which assumption on F one can recover for $F(M)$ statements that are true on $\mathcal{B}$.

In the literature, authors choose the presentation that suits best their problem and stick to it: people interested in Birkhoff normal forms or integrability issues deal with the moment map F itself, whereas in articles such as [LS10] or [Zun03], the authors are interested in results concerning the Lagrangian foliation. In particular, *they suppose that the fibers are connected.* As the original motivations for this work come from semi-classical theory, for us the given data is a moment map. We will often deal with the Lagrangian foliation, but we will always distinguish carefully between $\pi_{\mathcal{F}}$ and F, between $\mathcal{B}$ and $F(M)$. Showing that the fibers are connected is a recurring objective in this thesis.

One other motivation of this thesis is the classification of integrable systems, initiated by Delzant in the toric case (see Section 1.4). A classification of a given collection of objects means a choice of a set of morphisms between objects, so that we shall consider that two objects are the same to our purpose if there exists an isomorphism between them. The equivalence relation and equivalence class follow from this choice of morphisms. We have the following equivalence relation on integrable systems, which is called "weak equivalence" in [VN07]:

Definition 1.2.8. *For an integrable system F, we define the local commutant of* $\mathbf{f}$ *on an open* $\mathcal{U} \in M$*:*

$$\mathbf{C_f}(\mathcal{U}) = \{h \in \mathcal{C}^\infty(\mathcal{U} \to \mathbb{R}) \mid \{h, f_i\} = 0\}.$$

The local commutant is an Abelian Poisson algebra, and here also a Lie subalgebra. It is actually a sheaf. On a given open set $\mathcal{U}$*, two integrable systems* F *and* G *are said to be* equivalent*, and we note* $F \sim G$ *(or* $\mathbf{f} \sim \mathbf{g}$*), if* $\mathbf{f} \subset \mathbf{C_g}$*. This is an equivalence relation.*

Here again, the relation rely only on the Abelian algebra $\mathbf{f}$ and not on the moment map itself. We will see that for integrable systems, near regular points, we have

$$\mathbf{C_f}(\mathcal{U}) = \mathrm{Diff}(F(\mathcal{U})) \circ F \text{ , where } F \text{ is a moment map of } \mathbf{f}.$$

1.2.2 Near regular points

Here we suppose given an integrable system F.

Definition 1.2.9. *The rank* k_x *of an integrable system at a point* $m \in M$ *is the rank of* F *at* m*:* $k_\mathrm{x} = \dim(\mathrm{im}\,(dF(m)))$*. It is obvious that the rank depends only of* $\mathbf{f}$*.*

A point of F *is called* regular *if it is of maximal rank, that is, if* M^{2n} *and* $df_1 \wedge \cdots \wedge df_n(m) \neq 0$*. It is called* critical*, or* singular*, otherwise. It is called a* fixed point *if* $dF(m) = 0$*.*

The x in the definition of the rank symbolizes the fact that in the normal form theorem of Eliasson (Theorem 1.2.23), the rank is actually equal to the number of transverse components of the local model, which are notified by an X (see Section 1.2.3).

Now, with the local submersion theorem, we know that near a regular point, F can be linearized by a diffeomorphism. The next theorem tells us that it can be symplectically linearized near a regular point:

Theorem 1.2.10 (Darboux-Carathéodory)**.** *If* m *is a regular point, then there exists a neighborhood* $\mathcal{U}$ *of* m *and functions*

$$f_{k+1}, \ldots, f_n, u_1, \ldots, u_n \in \mathcal{C}^\infty(M \to \mathbb{R})$$

such that $\sum_{i=1}^n df_i \wedge du_i$ *is the canonical symplectic form on* $\mathbb{R}^{2n}$ *and*

$$\phi = (f_1, \ldots, f_n, u_1, \ldots, u_n) : (\mathcal{U}, \omega) \to (\mathbb{R}^{2n}, \sum_{i=1}^n df_i \wedge du_i)$$

is a symplectomorphism.

This theorem means that near a regular point of a moment map, we can take its components and complete it to get Darboux local coordinates.

An immediate consequence of the Darboux-Caratheodory theorem is that in a small enough neighborhood of a regular point, equivalence between $\mathbf{f}$ and $\mathbf{g}$ is equivalent to the existence of a local diffeomorphism U of $\mathbb{R}^k$ such that $F = U \circ G$. A lot of results in this thesis are formulated in terms of the existence of such an U near singular points, and its properties.

Darboux-Caratheodory theorem comforts the idea that *"locally, all integrable systems look the same"*. However, the leaves of the foliation given by the connected components of the level sets of the moment map are "semi-global" objects. So the question now is to see if there is a non-trivial semi-global theory.

A natural question would be to ask whether we can have a nice description of the foliation near a leaf with only regular points. But one cannot reasonably expect to have such a description without any hypothesis on the topology of the fibers. We will thus suppose that the moment map is proper, which implies that its fibers are compact. Of course an important case of a proper moment map is when we suppose M compact.

Assumption 1.2.11. *From now on, we will only consider integrable Hamiltonian systems on compact manifolds.*

Definition 1.2.12. *A value $c \in \operatorname{im}(F)$ is called* regular *if for all $m \in F^{-1}(c)$, m is a regular point. A* regular leaf *Λ is a connected component of a regular fiber.*

Definition 1.2.13. *Given a foliation on a topological manifold, a neighborhood $\mathcal{U}$ of a set is called* saturated *if for all p in $\mathcal{U}$, the leaf containing p is also in $\mathcal{U}$.*

Let's set $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{T}^r = (S^1)^r$. We are now ready to formulate the famous theorem of action-angle variables:

Theorem 1.2.14 (Liouville-Arnold-Mineur : "action-angle variables")**.** *Let F be a proper integrable Hamiltonian system, and c a regular value of F.*

Then, for a regular leaf $\Lambda_b \subseteq F^{-1}(c)$, there exists a saturated neighborhood $\mathcal{V}(\Lambda_b)$, a local diffeomorphism U of $\mathbb{R}^k$ and a symplectomorphism

$$\varphi : \operatorname{im}(\varphi) \subseteq (T^*\mathbb{T}^n, \omega_0) \to (\mathcal{V}(\Lambda_b), \omega) \subseteq M$$

such that:

- *φ sends Λ_b on the zero section.*
- *$\varphi^* F = F \circ \varphi = (I_1, \dots, I_n) = U(\xi_1, \dots, \xi_n)$.*

In other terms, there exists Darboux coordinates $(\boldsymbol{\theta}, \mathbf{I})$ with $\boldsymbol{\theta} \in \mathbb{T}^n$ and $\mathbf{I} \in \mathcal{B}(0, \eta)$ on a tubular neighborhood of Λ_b such that $\Lambda_b = \{I_1 = \cdots =$

$I_n = 0\}$. Geometrically, Liouville-Arnold-Mineur theorem straightens the Lagrangian foliation to the trivial n-torus fibration near a regular value.

The actions variables can be calculated by Mineur formula: since on a Lagrangian leaf Λ_b we have

$$i_{\Lambda_b}\omega = 0$$

by a theorem of Weinstein there exists a Liouville 1-form α such that $\omega = d\alpha$ on a tubular neighborhood of Λ_b. Then, if $(\gamma_1, \dots, \gamma_n)$ is the canonical basis of $H^1(\mathbb{T}^n)$, we have

$$I_i = \frac{1}{2\pi}\int_{\chi^*\gamma_i} \alpha \tag{1.4}$$

Rewriting Hamilton's equations 1.1 in action-angle coordinates, we get

$$(\text{Ham}) \iff \begin{cases} \dfrac{d\theta_i}{dt} = \dfrac{\partial H}{\partial I_i} = c_i \text{ constant } , \ i = 1, \dots, n \\ \dfrac{dI_i}{dt} = -\dfrac{\partial H}{\partial \theta_i} = 0 \ , \ i = 1, \dots, n \end{cases} \tag{1.5}$$

The Hamiltonian motion of a particule is restricted to a Lagrangian torus with constant speed: we say it is quasi-periodic.

1.2.3 Near critical points

A generic Hamiltonian system has critical points, and a lot of information can be extracted from the study of its critical points. The terminology we use here is very close to the one used by Zung in [Zun96]. As in the general setting of differential geometry, a lot of results can be obtained to describe the geometry of the moment map provided that the critical points verify some non-degeneracy condition.

Let p be a fixed point of an integrable system F. Since $dF(p) = 0$, in a Darboux chart the Hessians $\mathcal{H}(f_i)(p), i = 1, \dots, n$ are a subalgebra of the Poisson algebra of quadratic functions $(\mathcal{Q}(\mathbb{R}^{2n} \to \mathbb{R}), \{\cdot, \cdot\}_p)$.

Definition 1.2.15. *A fixed point p of F is called* non-degenerate *if the Hessians $\mathcal{H}(f_i)(p)$ defined above generate a Cartan subalgebra of $(\mathcal{Q}(\mathbb{R}^{2n} \to \mathbb{R}), \{\cdot, \cdot\}_p)$.*

Now, we consider a critical point of F of rank k_{x}. We may assume without loss of generality that $df_{n-k_{\mathrm{x}}+1} \wedge \cdots \wedge df_n \neq 0$. We can thus apply Darboux-Caratheodory to the sub-integrable system $\langle f_{n-k_{\mathrm{x}}+1}, \dots, f_n \rangle$: there exists a symplectomorphism $\varphi(\mathcal{U}, \omega) \to (\mathbb{R}^{2n}, \sum_{i=1}^n df'_i \wedge du_i)$ such that $f'_j = f_j - f_j(0)$ are canonical coordinates ξ_j for $j \geqslant n - k_{\mathrm{x}} + 1$. In these local coordinates, since the f_j are Poisson commuting, $f_1, \dots, f_{n-k_{\mathrm{x}}}$ do not depend of $x_{n-k_{\mathrm{x}}+1}, \dots, x_n$. In Darboux-Caratheodory, we can always suppose that $\varphi(m) = 0$. We define the function $g_j : \mathbb{R}^{2(n-k_{\mathrm{x}})} \to \mathbb{R}$, $1 \leqslant j \leqslant n - k_{\mathrm{x}}$ as $g_j(\bar{x}, \bar{\xi}) = f'_j(\bar{x}, 0, \bar{\xi}, 0)$.

Definition 1.2.16. *A critical point of rank k_{x} is called* non-degenerate *if for the g_j's defined above, the Hessians $\mathcal{H}(g_i)(p)$ generate a Cartan subalgebra of $(\mathcal{Q}(\mathbb{R}^{2(n-k_{\mathrm{x}})} \to \mathbb{R}), \{\cdot,\cdot\}_p)$. A Hamiltonian system is called non-degenerate if all its critical points are non-degenerate.*

We shall see in the section 1.3 that what we did above is just the explicite construction of the symplectic quotient with respect to the infinitesimal action of $\mathbb{R}^{k_{\mathrm{x}}}$ generated by $\langle f_{n-k_{\mathrm{x}}+1}, \dots, f_n\rangle$.

One last remark is that this definition of a non-degenerate point can be easily extended to regular points, which are hence always non-degenerate.

Assumption 1.2.17. *From now on, all systems we will consider have only non-degenerate critical points.*

Williamson index

In the article [Wil36], Williamson relied on the Lie algebra and Poisson algebra isomorphism $(\mathcal{Q}(\mathbb{R}^{2(n-k_{\mathrm{x}})} \to \mathbb{R}), \{\cdot,\cdot\}) \simeq (\mathfrak{sp}(2(n-k_{\mathrm{x}})), [\cdot,\cdot])$, to give the following classification theorem:

Theorem 1.2.18. [Williamson] *The Hessians one can compute at a non-degenerate critical point of an integrable system can be classified in 3 types:*

- *elliptic type (2×2 block): $\mathcal{H}_i = q_{\mathrm{e}}^{(i)} = x_i^2 + \xi_i^2$,*
- *hyperbolic type (2×2 block): $\mathcal{H}_i = q_{\mathrm{h}}^{(i)} = x_i\xi_i$,*
- *focus-focus type (4×4 block): $\begin{cases} \mathcal{H}_i = q_1^{(i)} = x_i\xi_i + x_{i+1}\xi_{i+1}, \\ \mathcal{H}_{i+1} = q_2^{(i)} = x_i\xi_{i+1} - x_{i+1}\xi_i. \end{cases}$*

We will characterize a non-degenerate critical point by the number of elliptic, hyperbolic and focus-focus blocks in its Williamson decomposition, plus its rank. We'll call the blocks associated to the rank *transversal* blocks and denote them with $X^{k_{\mathrm{x}}}$.

Definition 1.2.19. *We define the (generalized) Williamson type of a non-degenerate critical point as the quadruplet $\Bbbk = (k_{\mathrm{e}}, k_{\mathrm{h}}, k_{\mathrm{f}}, k_{\mathrm{x}}) \in \mathbb{N}^4$, where*

- *k_{e} is the number of elliptic blocks,*
- *k_{h} is the number of hyperbolic blocks,*
- *k_{f} is the number of focus-focus blocks,*
- *k_{x} is the number of transverse blocks.*

In the rest of the thesis, we'll speak of a point of Williamson type $\Bbbk$ to refer to a (possibly critical) non-degenerate point of Williamson type $\Bbbk$. Equivalently, we'll denote a point of Williamson type $\Bbbk$ with the obvious notation $FF^{k_{\mathrm{f}}} - E^{k_{\mathrm{e}}} - H^{k_{\mathrm{h}}} - X^{k_{\mathrm{x}}}$. As for now, we only work in the semi-toric case : $k_{\mathrm{h}} = 0$ and $k_{\mathrm{f}} = 0 or 1$. A point can also be regular if $k_{\mathrm{e}} = k_{\mathrm{h}} = k_{\mathrm{f}} = 0$ and $k_{\mathrm{x}} = n$. Indeed, we always have the following constraint over these coefficients:

$$k_{\mathrm{e}} + 2k_{\mathrm{f}} + k_{\mathrm{h}} + k_{\mathrm{x}} = n. \tag{1.6}$$

The Williamson type is by definition a symplectic invariant : if φ is a symplectomorphism, $\Bbbk(\varphi(m)) = \Bbbk(m)$.

We can define some structure on the sets of Williamson indices:

Definition 1.2.20. *The set $\mathcal{W}_0^n = \{\Bbbk \in \mathbb{N}^4 \mid k_\mathrm{e} + 2k_\mathrm{f} + k_\mathrm{h} + k_\mathrm{x} = n\}$ is the set of all possibles Williamson indices for integrable systems of n degrees of liberty. It is a graded partially-ordered set (or* poset*) for the order relation $\preccurlyeq$ defined by*

$$\Bbbk \preccurlyeq \Bbbk' \text{ if : } k_\mathrm{e} \geqslant k'_\mathrm{e}, k_\mathrm{f} \geqslant k'_\mathrm{f} \text{ and } k_\mathrm{h} \geqslant k'_\mathrm{h}$$

and for which the rank function is defined as

$$\begin{aligned} k_\mathrm{x} : \mathcal{W}_0^n &\to \mathbb{N} \\ \Bbbk &\to n - (k_\mathrm{e} + 2k_\mathrm{f} + k_\mathrm{h}) \end{aligned}$$

so that, if $\Bbbk$ covers $\Bbbk'$, that is,

$$\Bbbk \preccurlyeq \Bbbk' \text{ and } \nexists \ell \text{ s.t. } \Bbbk \preccurlyeq \ell \preccurlyeq \Bbbk'$$

then we have $k'_\mathrm{x} = k_\mathrm{x} + 1$. We also define the graded posets:

$$\mathcal{W}^n_{\preccurlyeq\Bbbk} = \{\Bbbk' \in \mathcal{W}_0^n \mid \Bbbk' \preccurlyeq \Bbbk\} \text{ and } \mathcal{W}^n_{\succcurlyeq\Bbbk} = \{\Bbbk' \in \mathcal{W}_0^n \mid \Bbbk' \succcurlyeq \Bbbk\}.$$

The proof that $\preccurlyeq$ is a partial order on $\mathcal{W}_0^n$ is straightforward. We can represent the poset by a directed acyclic graph with labeled vertices. The drawing rules are that for $\Bbbk, \Bbbk' \in \mathcal{W}_0^n$, line segments go upward every time $\Bbbk$ covers $\Bbbk'$. Segments can cross each other, but they must only touch their endpoints. Such a graph is called the *Hasse diagram* of the graded poset $\mathcal{W}_0^n$, and in the almost-toric case $k_\mathrm{h} = 0$ that interests us, it has a nice visual:

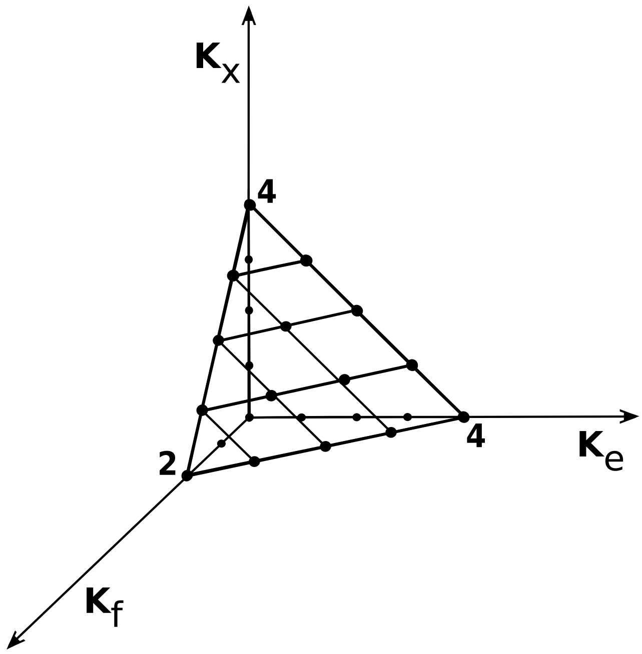

Figure 1.1: Hasse diagram of an almost-toric system

The poset $\mathcal{W}_0^n$ is the poset of all possible Willamson indices with the constraint (1.6), but for a given integrable system F, only a subset of it may appear. We simply define $\mathcal{W}_0^n(F)$ as the poset of Williamson indices such that there exists a $p \in M$ with that Williamson index for F. Again, the definition of this set only depends of $\mathbf{f}$.

The last thing we define here is the sheaf of sets given by the critical points of a fixed Williamson type $\Bbbk \in \mathcal{W}_0^n(F)$ for an integrable system F:

Definition 1.2.21.

$$CrP_{\Bbbk}^{F}(\mathcal{U}) = \{m \in \mathcal{U} \mid m \textit{ is a critical point of } F \textit{ of Williamson type } \Bbbk\}.$$

When the context makes it obvious, we ommit the mention to which system we are taking the critical points: $CrP_{\Bbbk}(\mathcal{U})$.

Linear model

We can define a linear model associated to a point of Williamson type $\Bbbk$:

Definition 1.2.22. *For a non-degenerate critical point p of Williamson type $\Bbbk$, we define the associated linear model as the Hamiltonian system $(L_{\Bbbk}, \omega_{\Bbbk}, \mathbf{q}_{\Bbbk})$, where :*

– *The manifold is*

$$L_{\Bbbk} = \left(\mathbb{R}^4\right)^{k_{\mathrm{f}}} \times \left(\mathbb{R}^2\right)^{k_{\mathrm{e}}} \times \left(\mathbb{R}^2\right)^{k_{\mathrm{h}}} \times T^*\mathbb{T}^{k_{\mathrm{x}}}.$$

– *The symplectic form is*

$$\begin{aligned}\omega_{\Bbbk} &= \omega_{\Bbbk}^{\mathrm{f}} + \omega_{\Bbbk}^{\mathrm{e}} + \omega_{\Bbbk}^{\mathrm{h}} + \omega_{\Bbbk}^{\mathrm{x}},\\ &= \sum_{i=1}^{k_{\mathrm{f}}} dx_{2i}^{\mathrm{f}} \wedge d\xi_{2i}^{\mathrm{f}} + dx_{2i-1}^{\mathrm{f}} \wedge d\xi_{2i-1}^{\mathrm{f}} + \sum_{i=1}^{k_{\mathrm{e}}} dx_i^{\mathrm{e}} \wedge d\xi_i^{\mathrm{e}}\\ &\quad + \sum_{i=1}^{k_{\mathrm{h}}} dx_i^{\mathrm{h}} \wedge d\xi_i^{\mathrm{h}} + \sum_{i=1}^{k_{\mathrm{x}}} d\theta_i \wedge dI_i.\end{aligned}$$

– *The integrable system is*

$$Q_{\Bbbk} = \left(q_1^1, q_1^2, \dots, q_{k_{\mathrm{f}}}^1, q_{k_{\mathrm{f}}}^2, q_1^{\mathrm{e}}, \dots, q_{k_{\mathrm{e}}}^{\mathrm{e}}, q_1^{\mathrm{h}}, \dots, q_{k_{\mathrm{h}}}^{\mathrm{h}}, I_1, \dots, I_{k_{\mathrm{x}}}\right).$$

with :
– $q_i^{\mathrm{e}} = (x_i^{\mathrm{e}})^2 + (\xi_i^{\mathrm{e}})^2$,
– $q_i^{\mathrm{h}} = x_i^{\mathrm{h}}\xi_i^{\mathrm{h}}$,
– $q_i^1 = x_{2i-1}^{\mathrm{f}}\xi_{2i-1}^{\mathrm{f}} + x_{2i}^{\mathrm{f}}\xi_{2i}^{\mathrm{f}}$ *and* $q_i^2 = x_{2i-1}^{\mathrm{f}}\xi_{2i}^{\mathrm{f}} - x_{2i}^{\mathrm{f}}\xi_{2i-1}^{\mathrm{f}}$.

We denote the associated Abelian algebra $\mathbf{q}_{\Bbbk} : L_{\Bbbk} \to \mathbb{R}^n$.

For shortness, we shall write $\mathbf{x}^- = (x_1^-, \dots, x_{k_-}^-)$ and $\boldsymbol{\xi}^- = (\xi_1^-, \dots, \xi_{k_-}^-)$, replacing the $-$ by f, e or h, and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{k_{\mathrm{x}}})$ and $\mathbf{I} = (I_1, \dots, I_{k_{\mathrm{x}}})$. When necessary, we also use the complex variables $z^{\mathrm{e}} = x^{\mathrm{e}} + i\xi^{\mathrm{e}}$, $z_h = x^{\mathrm{h}} + i\xi^{\mathrm{h}}$, $z_1^{\mathrm{f}} = x_1^{\mathrm{f}} + ix_2^{\mathrm{f}}$ and $z_2^{\mathrm{f}} = \xi_1^{\mathrm{f}} + i\xi_2^{\mathrm{f}}$. Note that here the complex variables are different in the focus-focus case than in the elliptic and hyperbolic case. Again, we use the compact notation $\mathbf{z}^-$, replacing $-$ by f, e or h.

We call *elementary* the (2×2)-elliptic, (2×2)-hyperbolic, and (4×4)-focus-focus blocks, as they are the simplest examples of non-degenerate singularities. We summarize in the array below some of their properties.

Type of block	Critical Levelsets	Expression of the flow in local coordinates
Elliptic	$\{q_e = 0\}$ is a point in $\mathbb{R}^2$	$\phi_{q_e}^t : z^{\mathrm{e}} \mapsto e^{it}z^{\mathrm{e}}$
Hyperbolic	$\{q_h = 0\}$ is the union of the lines $\{x^{\mathrm{h}} = 0\}$ and $\{\xi^{\mathrm{h}} = 0\}$ in $\mathbb{R}^2$	$\phi_{q_h}^t : (x^{\mathrm{h}}, \xi^{\mathrm{h}}) \mapsto (e^{-t}x^{\mathrm{h}}, e^t\xi^{\mathrm{h}})$
Focus-focus	$\{q_1 = q_2 = 0\}$ is the union of the planes $\{x_1 = x_2 = 0\}$ and $\{\xi_1 = \xi_2 = 0\}$ in $\mathbb{R}^4$	$\phi_{q_1}^t : (z_1^{\mathrm{f}}, z_2^{\mathrm{f}}) \mapsto (e^{-t}z_1^{\mathrm{f}}, e^tz_2^{\mathrm{f}})$ and $\phi_{q_2}^t : (z_1^{\mathrm{f}}, z_2^{\mathrm{f}}) \mapsto (e^{it}z_1^{\mathrm{f}}, e^{it}z_2^{\mathrm{f}})$

Table 1.1: Properties of elementary blocks

Below we give a first representation of the linear model near a focus-focus singularity. We can see immediately that the field X_{q_1} is along radial trajectories, that is, half-lines starting at the focus-focus critical points, while the field X_{q_2} is just the field of the rotation around the focus-focus critical points.

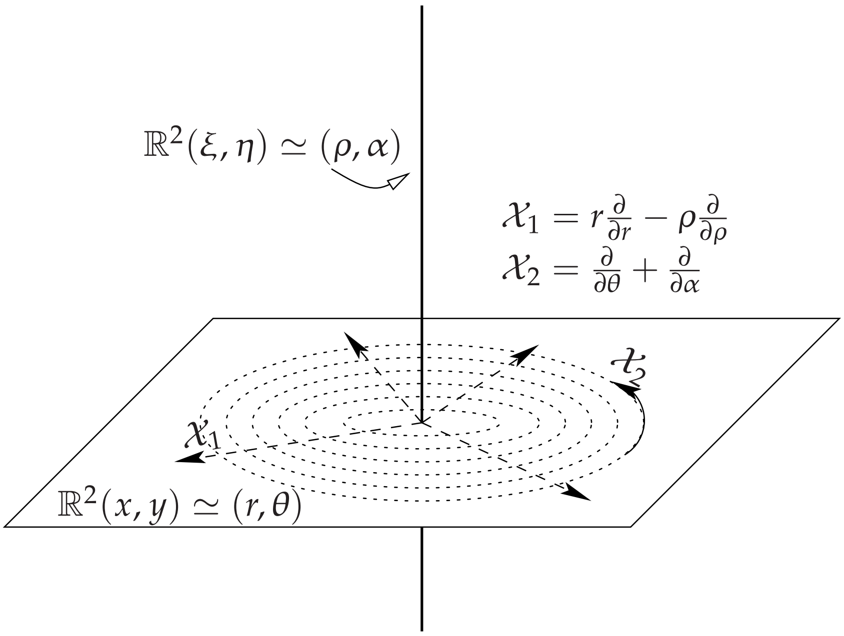

Figure 1.2: Linear model of focus-focus critical point

We then enounce the normal form theorem of Eliasson in its full generality. Its proof was the object of the thesis of Eliasson [Eli84], but only the elliptic case was published back then (see [Eli90]). Its proof in the focus-focus case is the object of Chapter 2.

Theorem 1.2.23 (Eliasson)**.** *For a critical point m of Williamson type $\mathbb{k}$ of an integrable system F, there exists a neighborhood $\mathcal{U}_m$ of m and a symplectomorphism $\chi : \mathcal{U}_m \to \chi(\mathcal{U}_m) \subset L_{\mathbb{k}}$ such that*

$$\chi^* F \sim Q_{\mathbb{k}}.$$

In particular, if $k_{\mathrm{h}} = 0$, there exists a local diffeomorphism $G : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\chi^* F = G \circ Q_{\mathbb{k}}.$$

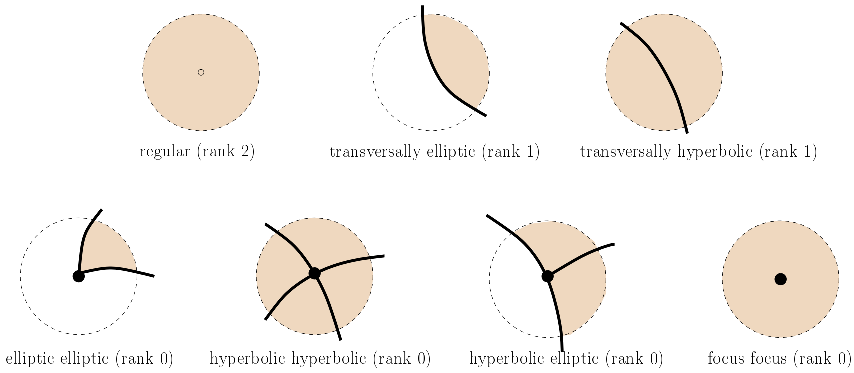

Figure 1.3: Local models of critical points in dimension $2n = 4$. The colored part corresponds to the image of the moment map, and the black line to critical points of various ranks.

Eliasson normal form theorem gives also a local model of the image of the moment map near a critical point of a given Williamson type. In two dimensions, it is well known that elliptic-elliptic, hyperbolic-hyperbolic, or elliptic-hyperbolic critical points are isolated fixed points, as well as focus-focus points. We also have that transversally elliptic, hyperbolic points always come as 1-parameter families of critical points. Hence, we have the picture above for local models of critical points in dimension $2n = 4$.

1.3 Symplectic actions of Lie groups

Another point of view to understand how symplectic geometry provides a good framework for the equations of mechanics is to see the conservation laws as actions of Lie group on the space of motion that is given by M. We start with an equivariant version of 1.1.6 :

Theorem 1.3.1 (Equivariant Darboux theorem). *Let M be a symplectic manifold, K a compact subgroup of symplectomorphisms, $p \in M$ a fixed point for K. There exists a K-equivariant symplectomorphism of a neighborhood of p to a neighborhood of $0 \in T_pM$.*

The first example is when we have a symplectic field. Its flows gives a 1-parameter group of symplectomorphisms : it is an action of the Lie group $\mathbb{R}^1$. More generally, let G be a connected Lie group acting on a manifold M, and let $\mathfrak{g}$ denote its Lie algebra. Remember that $\mathfrak{g} := \chi(M)^G \simeq T_eG$: an element of T_eG gives a vector field on G that is left-variant by G. The Lie bracket is defined by $[X, Y] := (X_M Y_M - Y_M X_M)_e$. One has a natural morphism :

$$\begin{aligned} &\mathfrak{g} \to \mathcal{X}^1(M) \\ &X \mapsto X_M : \left(p \mapsto X_M(p) = \frac{d}{dt}(exp(tX) \cdot p)\bigg|_{t=0} \right). \end{aligned}$$

We can give now the first definition:

Definition 1.3.2. *One says that the action of G on M is* symplectic *if it preserves ω.*

1.3.1 Hamiltonian actions and moment map

Among symplectic actions of Lie groups, we can define the Hamiltonian actions, and their associated moment map. Our main reference for all this subject is [Sou97].

Definition 1.3.3. *A symplectic group action is called* Hamiltonian *if, for all $X \in \mathfrak{g}$, the vector field X_M is Hamiltonian. If so, one then has a linear map:*

$$\begin{aligned} \tau : \mathfrak{g} &\to \mathcal{C}^\infty(M \to \mathbb{R}) \\ X &\mapsto H^X. \end{aligned}$$

In general, this morphism does not behave correctly with respect to the Lie structure. We have however the following result for semi-simple Lie groups:

Proposition 1.3.4. *For G a semi-simple Lie group, we can define the morphism $\tau : (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \to (\mathcal{C}^\infty(M \to \mathbb{R}), \{\cdot, \cdot\})$ such that it is a Lie algebra morphism.*

Proof. Using Poisson formula (see Prop. 1.2.2), $\{H^X, H^Y\}$ is a Hamiltonian for $[X, Y]$, so there exists a constant c (depending of X and Y) such that

$$\{H^X, H^Y\} - H^{[X,Y]} = c(X, Y).$$

The constant $c(X,Y)$ can be seen as a 2-cochain. It is a 2-cocycle, and since G is semi-simple, we know by a lemma of Whitehead that it is a 2-coboundary: $c(X,Y) = l([X,Y])$ with l a 1-cochain. We redefine now τ as $H^X + l$, it is a Lie algebra morphism.

$\square$

We axiomatize this last property in the following definition:

Definition 1.3.5. *A symplectic action $G \circlearrowright M$ of a Lie group is called* Poisson *if there exists a Lie algebra morphism*

$$H^- : (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}) \to (\mathcal{C}^\infty(M \to \mathbb{R}), \{\cdot,\cdot\})$$

which lifts to the natural map

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r] & (\mathcal{C}^\infty(M \to \mathbb{R}), \{\cdot,\cdot\}) \arrow[r, "J\nabla"] & (\mathcal{X}^1(M), [\cdot,\cdot]) . \\ & & (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}) \arrow[u, dotted, "H^-"] \arrow[ur, "\exp"'] & \end{tikzcd}$$

For a Poisson action, one can define the moment map *of the Poisson action*

$$\begin{aligned} J : M &\to \mathfrak{g}^* \\ p &\mapsto (X \mapsto H^X(p)) \end{aligned}$$

the dual map of H^-. Now, having such a map makes it easy to compute Hamiltonians : for $X \in \mathfrak{g}$, $\langle J(p), X\rangle$ is a Hamiltonian for X.

The first action of a group to consider is on itself :

Definition 1.3.6. *The* **adjoint** *action Ad_g of $g \in G$ on $\mathfrak{g}$, is the derivative at e of the map $C_g : a \mapsto gag^{-1}$. It is an invertible linear map of $\mathfrak{g}$, that defines a left group action on it : $g \cdot X := Ad_g X$. We can also write*

$$Ad_g(X) = \left.\frac{d}{dt}\right|_{t=0} g \exp(tY) g^{-1}.$$

The **coadjoint** *action is defined by duality. First, for $\lambda \in \mathfrak{g}^*$ the element $Ad_g^*\lambda$ is completely defined by*

$$\forall X \in \mathfrak{g}\ ,\ \left(Ad_g^*\lambda\right)(X) = \lambda(Ad_g X).$$

Next, we define a left group action of G on $\mathfrak{g}^*$: $g \cdot \lambda := Ad_{g^{-1}}^*\lambda$. This is the coadjoint action.

We denote the differential of the adjoint representation by ad. There is the important particular case when the Lie group is a matrix group. In that case, for $X, Y \in \mathfrak{g}$

$$ad_X(Y) = \frac{d}{dt} Ad_{exp(tX)} Y \Big|_{t=0} = [X, Y].$$

Proposition 1.3.7. *Let $G \circlearrowleft M$ be a Poisson action of a connected Lie group. We have that J is equivariant for some affine action on $\mathfrak{g}^*$ whose linear part is Ad^*. Moreover, if G is compact, then J is exactly Ad^*-equivariant.*

Proof. For $X, Y \in \mathfrak{g}$, $p \in M$,

$$\begin{aligned}
\langle \frac{d}{dt} J(\exp(tX) \cdot p)|_{t=0}, Y\rangle &= \mathcal{L}_{X_M} \langle J(p), Y\rangle = X_M.H^Y = dH^Y(X_M) \\
&= \omega_p(X_M(p), Y_M(p)) = H^{[X,Y]}(p) = \langle J(p), [X, Y]\rangle \\
&= \langle ad^*_X J(p), Y\rangle = \langle \frac{d}{dt} \left(Ad^*_{\exp(tX)} J(p)\right)|_{t=0}, Y\rangle.
\end{aligned}$$

This holds for all $Y \in \mathfrak{g}$ so we have

$$\frac{d}{dt} \left[Ad^*_{\exp(tX)} J(p) - J(\exp(tX) \cdot p)\right]_{t=0} = 0 \ ,$$

and since G is connected, exp is surjective : the equivariance is affine for all $g \in G$. If we suppose in addition that G is compact, then with the Haar measure associated to $G \circlearrowleft M$, one can construct the center of mass of M for the action of G, which is a fixed point. The existence of such a fixed point implies the nullity of the constant part of the affine action.

□

This property motivates the following definition:

Definition 1.3.8. *The action $G \circlearrowleft M$ is strongly Hamiltonian if it is Poisson and J is Ad^*-equivariant.*

So, for a connected group we have the following implications

Strongly Hamiltonian $\Rightarrow$ Poisson $\Rightarrow$ Hamiltonian $\Rightarrow$ Symplectic.

In our particular setting of (sub)-integrable Hamiltonian systems, the group actions will always be strongly Hamiltonian, even though the groups are not always compact.

Example 1.3.9 (linear action of the torus)**.** *Firstly, let's take an action of a circle S^1 on E a Hermitian vector space of complex dimension n. We then* *know there exists an orthogonal basis of E and integers $a_1, \ldots, a_d \in \mathbb{Z}$ such that*

$$e^{i\theta} \cdot (z_1, \ldots, z_n) = (e^{ia_1\theta} z_1, \ldots, e^{ia_n\theta} z_n).$$

Thus, the vector field induced by a $\theta \in \mathfrak{g}^*$ admits $\frac{1}{2}\sum_{i=1}^n a_i|z_i|^2$ as a Hamiltonian. Hence a moment map for this action is

$$J(z_1, \ldots, z_n) = \frac{1}{2}\sum_{i=1}^n a_i|z_i|^2.$$

If we now have a torus $\mathbb{T}^d$ $(d \leqslant n)$ acting on E, there exists a lattice $L \subset \mathfrak{t}$ such that $X \in L$ if and only if $\forall p \in M$, $\exp(X) \cdot p = p$. It dual lattice $\Lambda \subset \mathfrak{t}^*$ is called the weight lattice. A unitary representation of $\mathbb{T}^d$ of dimension 1 *(i.e: an action of $\mathbb{T}^d$ on a Hermitian vector space of complex dimension* 1*) is* $\exp(\boldsymbol{\theta}) \cdot z = e^{i\langle\xi,\boldsymbol{\theta}\rangle} z$*, where ξ is a primary vector of Λ, and the moment map is $J(z) = \xi|z|^2$. When we have an action on a space E of complex dimension n, we can decompose E into d invariant subspaces, and compute the resulting moment map*

$$J(z_1, \ldots, z_n) = \sum_{i=1}^d \xi_i|z_i|^2$$

The vectors $\boldsymbol{\xi}_j = (\xi_{j,1}, \ldots, \xi_{j,n})$ then form a basis of Λ : this representation is the weight representation.

A good training for graduate students is to make the study of the Galileo group action on $\mathbb{R}^4$. Such study is made in [Sou64]. In this study appears an obstruction to equivariance (a cohomology class) that can be interpretated as the inertial mass of the object.

1.3.2 Moment map and stratification

In this section, we will always consider a strongly Hamiltonian action of a compact group G. It defines two stratifications of the symplectic manifold M^{2n} : one by the rank of J, the other by the dimension of the orbits. We will show that these two actually coincide, but first we need a definition of a stratified space. We define it in the setting of differential spaces, where the decomposed space is not necessarily a manifold. We need to do so as we shall deal with such pathological spaces later in Chapter 5. Presentation and examples in this section are inspired from [SL91] and [Pfl01].

Differential spaces

Definition 1.3.10. *Let* $\mathbf{C}$ *be a subalgebra of the algebra of continuous functions* $f : X \to \mathbb{R}$*. We say that* $\mathbf{C}$ *is locally detectable if a function* $h : X \to \mathbb{R}$ *is contained in* $\mathbf{C}$ *if and only if for all* $x \in X$ *there is an open neighbourhood* U *of* x *and* $g \in \mathbf{C}$ *such that* $h|_U = g|_U$*.*

We can see that a locally detectable algebra is equivalent to a subsheaf of the sheaf of continuous functions. We now define differential spaces.

Definition 1.3.11. *A differential space is a pair* $(X, \mathbf{C})$*, where* X *is a topological space and a locally detectable subalgebra* $\mathbf{C} \subset \mathcal{C}^0(X, \mathbb{R})$ *(or equivalently a space* X *together with a subsheaf of the sheaf of continuous functions on* X*) satisfying the condition:*

$$\forall f_1, \ldots, f_k \in \mathbf{C} \text{ and } g \in \mathcal{C}^\infty(\mathbb{R}^k \to \mathbb{R}) \ , \ g \circ (f_1, \ldots, f_k) \in \mathbf{C}.$$

This condition is clearly required in order to construct new elements of $\mathbf{C}$ by composition with smooth maps and it holds for smooth manifolds by the chain rule.

Before giving the definition of a stratified differential space, we will give the definition of a manifold with the language of differential spaces:

Definition 1.3.12. *A* k*-dimensional smooth manifold is a differential space* $(M, \mathbf{C})$ *where* M *is a Hausdorff space with a countable basis of its topology, such that for each* $x \in M$ *there is an open neighbourhood* $U \subset M$*, an open subset* $V \subset \mathbb{R}^k$ *and a differential space isomorphism (i.e. : a sheaf isomorphism):*

$$\varphi : (V, \mathcal{C}^\infty(V)) \to (U, \mathbf{C}(U)).$$

That is : a k-dimensional smooth manifold is a differential space which is locally isomorphic as a differential space to $(\mathbb{R}^k, \mathcal{C}^\infty(\mathbb{R}^k))$; here, φ is the chart. Thus, for a manifold, we shall write $\mathcal{C}^\infty(M)$ instead of $\mathbf{C}$.

Let's restate now the definition of the germ of a function in the context of a general differential space. We define an equivalence relation on $\mathbf{C}$ by setting $f \sim g$ if and only if there is an open neighbourhood V of x such that $f|_V = g|_V$. We call the **germ** of f at x the equivalence class represented by f and denote it by $[f]_x$. The stalk at x, that is, the space of all germs at a point x is noted $\mathbf{C}_x$. For a general differential space $(X, \mathbf{C})$ and a point x in X we can define, as in the case of manifolds, the tangent space at x as the vector space of all derivations of germs at x.

Definition 1.3.13. *The dimension of a differential space is the maximal* n *such that there exists an* $x \in X$ *with* $\dim(T_x X) = n$*.*

Stratified space

A stratified manifold is often defined simply as a filtration of a manifold by closed submanifolds that glue together nicely with respect to their respective differential structure. In our definition we both relax the manifold assumption and the desired properties of the underlying ordering set.

Definition 1.3.14. *Given a poset $(\mathcal{I}, \preccurlyeq)$, a $\mathcal{I}$-decomposition of a Hausdorff and paracompact space X is a locally finite collection $\mathcal{S} = (S_i)_{i\in\mathcal{I}}$ of disjoint locally closed manifolds such that*

i. $X = \bigsqcup_{i\in\mathcal{I}} S_i$,

ii. Frontier condition : $\mathcal{S}$ is a poset for the order relation $(S_i \leq S_i')$ if and only if $\left(S_i \subseteq \bar{S}_i'\right)$ called the frontier *order, and the map*

$$\begin{aligned}\mathbf{S} : (\mathcal{I}, \preccurlyeq) &\to (\mathcal{S}, \leq)\\ i &\mapsto S_i\end{aligned}$$

is an increasing map.

We call the space X an $\mathcal{I}$-decomposed space. The S_i's are called pieces of its decomposition. The set $\Sigma_r = \bigcup_{i\preccurlyeq r} S_i$ is called the r-skeleton of the decomposition.

If $S_i < S_i'$ (that is, $S_i \leq S_i'$ and $S_i \neq S_i'$), we say that S_i is **incident** to S_i' or that it is a **boundary** piece of S_i', and that S_i' is **excident** to, or is a **parent** piece of S_i.

Definition 1.3.15. *Given a decomposition $\mathcal{S}$ of X, we define the* **depth** *$dp_X(S)$ of a piece $S \in \mathcal{S}$ as*

$$dp_X(S) := \sup\{k \in \mathbb{N} \mid \exists S_0, S_1, \ldots, S_k \in \mathcal{S} \text{ with } S = S_0 < S_1 < \cdots < S_k\}.$$

We then define the depth of a stratified differential space as

$$\operatorname{depth}(X) = \sup_{i\in\mathcal{I}} dp_X(S_i).$$

For a differential space, there is a natural decomposition :

$$S_i := \{x \in X \mid \dim(T_xX) = i\}.$$

For an n-dimensional smooth manifold M we have that $S_n = M$ and all other strata are empty.

Given a $\mathcal{I}$-decomposed differential space, we define the following equivalence relation $\sim_{S_i}$ on $\mathbf{C}$: $f_1, f_2 \in \mathbf{C}$, $f_1 \sim_{S_i} f_2$ if $(f_1)|_{S_i} = (f_2)|_{S_i}$. We write $f\rfloor_{S_i}$ for the equivalence class and $\mathbf{C}(S_i) := \mathbf{C}/_{\sim_{S_i}}$. Of course, $\mathbf{C}(S_i)$ is a sheaf on S_i. Note that such an equivalence relation can be defined on any union of S_i.

Stratified spaces are a generalization of manifolds: they are differential spaces stratified by manifolds, with extra assumptions on $\mathbf{C}$. They are defined by induction on the dimension (cf. condition 2.).

Definition 1.3.16. *A non-empty n-dimensional* stradispace *(for* **stra***tified* **di***fferential* **space***) is a $\mathcal{I}$-decomposed differential space $(X, \mathbf{C}, \mathcal{S})$ such that:*

1. *For all $i \in \mathcal{I}$, $(S_i, \mathbf{C}(S_i))$ is a smooth manifold of dimension $\dim(S_i)$. We shall write $\mathcal{C}^\infty(S_i)$ for $\mathbf{C}(S_i)$.*
2. *Splitting condition: if $x \in S_i$, for a neighborhood U_x of x in X there exists a disk $\mathbf{D}^i \subset \mathbb{R}^i$ and a cone $\mathcal{C}(L)$ over a $n-i-1$-dimensional stradispace L, such that there exists an homeomorphism between $(U_x, \mathbf{C}(U_x))$ and $(\mathbf{D}^i \times \mathcal{C}(L), \mathcal{C}^\infty(\mathbf{D}^i \to \mathbb{R}) \times \mathbf{C}(\mathcal{C}(L)))$ that preserves the decomposition.*
3. *Point separation: For each $x, y \in X$ there exists a function $\rho \in \mathbf{C}$ such that $\rho(x) = 0$ and $\rho(y) \neq 0$.*

We say equivalently that $\mathcal{S}$ is a stratification *of the space $(X, \mathbf{C})$.*

Remark 1.3.17. *Note that in condition 2., the differential structure on the cone is not fixed by the structure on the strata. This is a difference between stratified differential spaces and the stratifolds developped by Matthias Kreck (see [Kre10]). Here, to have a stratifold, we shall ask for a* sheaf *isomorphism in the splitting condition. As a result, the algebra of smooth functions on a cone are locally constant near the singular point.*

The example to remember is the following:

Example 1.3.18. *Let G be a compact Lie group acting on a smooth manifold M. For a subgroup H of G denote by $M_{(H)}$ the set of all points whose stabilizer is conjugate to H. $M_{(H)}$ will be the stratum of M of* orbit type *(H), and the indexing set $\mathcal{I}$ will be the set of all possible stabilizer subgroups modulo the conjugacy relation: $H \sim K \iff (\exists g \in G \text{ with } gHg^{-1} = K)$.*

The name orbit-type *comes from the fact that for a compact Lie group G, we have that G/G_x and $G \cdot x$ are diffeomorphic manifolds and that if x and y are in the same orbit, then G_x and G_y are conjugate subgroups of G. Note however that G/G_x is not a group in general.*

The ordering is by reverse subconjugacy : the class of H is "bigger" than the class of K, $(H) \succcurlyeq (K)$, if and only if there exists $g \in G$ with $H \subseteq gKg^{-1}$. Here M is a $\mathcal{I}$-stratified manifold.

If we now take the quotient M/G, it is not a manifold, but it is a $\mathcal{I}$-decomposed space. Its canonical differential structure will be the smallest subalgebra $\mathbf{C}$ of $\mathcal{C}^0(M/G)$ that makes the orbit map $\pi : (M, \mathcal{C}^\infty(M) \to (M/G, \mathbf{C})$ smooth, i.e.

$$\mathcal{C}^\infty(M/G) = \{f : M/G \to \mathbb{R} \mid f \circ \pi \text{ is smooth } \}.$$

We have that $\mathcal{C}^\infty(M/G)$ is isomorphic to the space $\mathcal{C}^\infty(M)^G$, the space of G-invariant smooth functions on M.

We get other interesting examples of stratified manifolds or differential spaces by taking coarser or finer order relations : in the previous example, we can take a finer ordering by keeping the distinction between two conjugate groups. The ordering is thus simply the inclusion. On the contrary, we can make an equivalence class of all orbits that have a stabilizer of the same dimension. This amounts to consider all orbits of the same dimension as a single stratum, and it is the order relation we consider now.

Rank stratification and orbit stratification

We have the fundamental result concerning the moment map

Proposition 1.3.19. *For $p \in M$, the orbit $G \cdot p$ of p under $G \circlearrowleft M$ is an embedded submanifold. On each point, the tangent bundle is the symplectic orthogonal to the kernel of the moment map*

$$\ker(T_p J) = (T_p G \cdot p)^{\perp\omega}.$$

Proof. The orbit $G \cdot p$ is an embedded submanifold because G is a compact Lie group. For a $X \in \mathfrak{g}^*$ we have $\langle J(p), X \rangle = H^X(p)$. We differentiate this equation with respect to p : $\langle T_p J, X \rangle = dH^X(p)$. So, for $Y \in \ker(T_p J)$, we have that $dH^X_p(Y) = 0$, that is, $\omega_p(X_M(p), Y) = 0$ for all $X_M \in \mathcal{X}^1_{\mathfrak{g}}(M) = T_p(G \cdot p)$. $\square$

Corollary 1.3.20. *If G is connected, the stratification of M by the rank of J coincides with the stratification by the dimension of the orbits.*

These two statements are actually very important for the study of Hamiltonian actions of general Lie groups : in general, the orbit stratification is extremely diffcult to deal with, but in the case of Hamiltonian actions, we can study it through its moment map. We also have the following corollary:

Corollary 1.3.21. *The non-empty strata given by a strongly Hamiltonian action of a compact connected group $G \circlearrowleft M$ are symplectic manifolds.*

Proof. We can always linearize the group action near a critical point of the moment map. With Example 1.3.9, we get the desired result. $\square$

In the case of (semi-toric) integrable systems, we will prove in Chapter 4 results similar to Corollaries 1.3.20 and 1.3.21 but for a non-compact Lie group, and with a finer stratification (a stratification by the Williamson type).

1.3.3 Symplectic structure on co-adjoints orbits

In the 70's, Kirillov, Kostant and Souriau gave a recipe to build from a Lie group a symplectic manifold and a strongly Hamiltonian action on it.

Theorem 1.3.22 (Kirillov, Kostant, Souriau). *Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ its dual. On each coadjoint orbit $\mathcal{O}_{\mu_0}$, there exists a unique invariant symplectic structure so that the action of G is strongly Hamiltonian with moment map $J : \mathcal{O}_{\mu_0} \hookrightarrow \mathfrak{g}^*$.*

Proof. We fix a $\mu_0 \in \mathfrak{g}^*$ and construct the symplectic form ω_0 for $\mathcal{O}_{\mu_0}$. For a $\mu \in \mathfrak{g}^*$, we define the isotropy subgroup $G_\mu := \{g \in G \mid Ad_g^* \mu = \mu\}$, and $\mathfrak{g}_\mu := \{X \in \mathfrak{g} \mid \mathrm{ad}_X^* \mu = 0\}$ its Lie algebra. We also define

$$\mathcal{O}_\mu := \{Ad_g^* \mu, g \in G\}$$

the co-adjoint orbit of G in $\mathfrak{g}$.

On G, we define the Maurer-Cartan 1-form Θ, which takes its values in the fixed space $\mathfrak{g}$. It is the unique 1-form (we will admit this fact) that is invariant by the left action of G on itself and such that, for $Z \in \mathfrak{g}$, $\tilde{Z} \in \Gamma(TG)$ the associated left invariant vector field, we have $\Theta(\tilde{Z}) = Z$. We then associate to each $\eta \in \mathfrak{g}^*$ a 1-form $\tilde{\eta} = \eta \cdot \Theta \in \Gamma(T^*G)$.

The tangent space of $\mathcal{O}_\mu$ at μ satisfies

$$T_\mu \mathcal{O}_\mu = \{ad_X^* \mu, X \in \mathfrak{g}\} \ , \ \text{ so } \ T_\mu \mathcal{O}_\mu \cong \mathfrak{g}/\mathfrak{g}_\mu.$$

This identification allows us to define on $\mathcal{O}_{\mu_0}$ the 2-form ω defined by $\sigma = d\tilde{\mu}_0 = \pi^* \omega$ where

$$\begin{aligned} \pi : G &\to \mathcal{O}_{\mu_0} \\ g &\mapsto \mathrm{Ad}_{g^{-1}}^* \mu_0 \ . \end{aligned}$$

At the point $\mu = \mathrm{Ad}_{g^{-1}}^* \mu_0 \in \mathcal{O}_{\mu_0}$, the symplectic form is

$$\omega_\mu(\tilde{X}, \tilde{Y}) = \mu([X, Y]) = \mathrm{ad}_X^* \mu(Y) = -\mathrm{ad}_Y^* \mu(X).$$

Here the right side of the equation, X and Y are taken as elements of $\mathfrak{g}/\mathfrak{g}_\mu$. We can verify that ω is a 2-form, non-degenerate. It is G-invariant:

$$\begin{aligned} \text{For } \mu \in \mathcal{O}_{\mu_0} \ , \ (g_* \omega)_\mu(\tilde{X}, \tilde{Y}) &= (g \cdot \mu)([g \cdot X, g \cdot Y]) = \mathrm{Ad}_{g^{-1}}^* \mu(\mathrm{Ad}_g [X, Y]) \\ &= \mu(\mathrm{Ad}_{g^{-1}} \mathrm{Ad}_g [X, Y]) = \omega_\mu(\tilde{X}, \tilde{Y}) \ . \end{aligned}$$

Let us check it is closed. Since $\sigma = d\tilde{\mu}_0$, we have that $\sigma(\tilde{X}_g, \tilde{Y}_g) = \mu_0 \cdot d\Theta(\tilde{X}, \tilde{Y}) = \mu_0 \cdot [\Theta(\tilde{X}_g), \Theta(\tilde{Y}_g)]$, and hence,

$$\begin{aligned} \sigma(\tilde{X}_g, \tilde{Y}_g) &= \mu_0 \cdot [\Theta(\tilde{X}_g), \Theta(\tilde{Y}_g)] \\ &= \mu_0 \cdot [\mathrm{Ad}_{g^{-1}} X, \mathrm{Ad}_{g^{-1}} Y] \\ &= \mu_0 \cdot \mathrm{Ad}_{g-1} [X, Y] \\ &= \underbrace{\mathrm{Ad}_{g^{-1}}^* \mu_0}_{=\pi(g)} \cdot [X, Y] \ . \end{aligned}$$

We have that $\pi^* \omega_\mu = d\tilde{\mu}$ is an exact 2-form. Since π is an immersion, ω is closed. Last thing to check is that $J = id_{\mathfrak{g}^*}$, and it is straightforward. $\square$

The converse statement that any symplectic G-manifold is locally isomorphic to a coadjoint orbit of (a central extension of) G is also true but much trickier to show. Our point here is just to give insights into this technique, one of the major techniques used in the area of integrable systems.

1.3.4 Symplectic quotient

For all the section we will consider a strongly Hamiltonian action of a compact group K with moment map $J : M \to \mathfrak{K}^*$. Let $\mu \in \operatorname{im}(J)$ be a regular value of J. One would like to define a quotient of M by the action of K which respects the symplectic structure, a *symplectic quotient*. Remember that J is Ad^*-equivariant. We still have $K_\mu := \{k \in K \mid Ad_k^*\mu = \mu\}$ and $\mathfrak{K}_\mu := \{k \in \mathfrak{K} \mid ad_k^*\mu = 0\}$.

Lemma 1.3.23. *For $p \in M$, we have the equality of vector spaces*

$$T_p(K_\mu \cdot p) = (\mathfrak{K}_\mu) \cdot p = \ker(\omega|_{T_pJ^{-1}(\mu)}).$$

That is, the leaves of the foliation defined by $\ker(i^*_{TJ^{-1}(\mu)}\omega)$ are the orbits of K_μ.

Proof. As K is compact, connected, K_μ is compact, connected. This highly non-trivial fact is actually the statement 1. of Theorem 1.4.9.

The fiber $J^{-1}(\mu)$ is a submanifold of M which is globally invariant by K_μ: for $p \in J^{-1}(\mu)$, $k \in K_\mu$, $J(k \cdot p) = Ad_k^* J(p) = Ad_k^*\mu = \mu$. Then we have $T_pJ^{-1}(\mu) = \ker(T_pJ)$: when moving along the directions where $T_pJ \equiv 0$, we stay on the level set $J^{-1}(\mu)$. Then, we have, by Property 1.3.19,

$$\ker(\omega_{T_pJ^{-1}(\mu)}) := \ker(T_pJ)^\perp \cap \ker(T_pJ) = T_p(J^{-1}(\mu)) \cap T_p(K \cdot p).$$

This intersection is equal to $\mathfrak{K}_\mu \cdot p$: if $k \cdot p \in T_pJ^{-1}(\mu)$ then $k \in \mathfrak{K}_\mu$, the stabilizer of μ in $\mathfrak{K}$ under ad^*.

□

We have seen with Corollary 1.3.20 that the stratification by the rank of the orbits and by the rank of J coincide. Lemma 1.3.23 lets us hope that we'll be able to define our symplectic quotient on the level sets of J.

The problem is that K_μ acts only locally freely. A priori, there can be points whose isotropy group is non-trivial : the quotient has singular points. So, in order to define "nice" symplectic quotients, one must assume that K_μ acts freely on $J^{-1}(\mu)$:

Proposition 1.3.24. *If the action of K_μ is free on $J^{-1}(\mu)$, there exists a unique symplectic structure ω_μ on $M_\mu = J^{-1}(\mu)/K_\mu$ whose pullback by the orbit map to $J^{-1}(\mu)$ is $i^*_{TJ^{-1}(\mu)}\omega$.*

In the general case, the quotient space is the quotient of a manifold by a finite group (the isotropy group K_μ): it is an orbifold. With care, we can define on it the quotient symplectic structure. More generally, many definitions and results can be extended from the symplectic manifold to the symplectic orbifold setting, but the operation is never straightforward (see [LT97]).

The collection $(M_\mu, \omega_\mu)_{\mu \in J(M)}$ is our symplectic quotient. It is noted $M/\!\!/G$. There exists a "shifting trick" due to Marsden & Weinstein that allows us to take 0 for the symplectic quotient $M/\!\!/G = J^{-1}(0)/K_0$, and assume that 0 is a regular value of J.

The case of a μ that is not a regular value of J has been studied in [SL91]. The quotient space is no longer a manifold but the orbit-type stratification of M descends on the quotient. Each quotient stratum turns out to have a canonical symplectic structure. The smooth structure $\mathbf{C}(M/\!\!/G)$ is isomorphic to $\mathcal{C}^\infty(M)^G/I^G$, where I^G is the ideal of invariant functions vanishing on $J^{-1}(\mu)$. It inherits a Poisson algebra structure from $\mathcal{C}^\infty(M)$ that is compatible with the symplectic forms on the strata of $\mathbf{C}(M/\!\!/G)$. We can axiomatize these properties to define $M/\!\!/G$ as a *symplectic stradispace*.

Definition 1.3.25. *A symplectic stradispace is a stradispace such that :*

1. *Each stratum ($S_i, \mathbf{C}(S_i)$) is a symplectic manifold.*
2. *The differential structure* $\mathbf{C}$ *on* X *is a Poisson algebra.*
3. *The embedding* $S_i \hookrightarrow X$ *is Poisson.*

Symplectic quotient is a technique widely used in all the areas of symplectic geometry. As an example, we can redefine integrability of a Hamiltonian action by saying that a Hamiltonian action $G \circlearrowleft M$ is **integrable** if its symplectic quotient is reduced to a point.

1.4 Toric systems

Since (sub)-integrable Hamiltonian systems on compact manifolds have complete flows, they can be seen as a strongly Hamiltonian action of $\mathbb{R}^k$, seen as a connected Abelian Lie group.

Definition 1.4.1. *A (sub-)integrable system $F : M^{2n} \to \mathbb{R}^k$ is called k-*toric *if F is a moment map for a (global) Hamiltonian $\mathbb{T}^k$-action. In the case of an integrable system, if $k = n$ we simply say* toric.

That is to say, a (sub-)integrable system is toric if all of its components are periodic. Remembering that F is a basis of the Abelian Poisson algebra $\mathbf{f}$, we have with our definition above that if we take a toric F, another moment map $\tilde{F}$ for $\mathbf{f}$ is toric if it is obtained from F by left-composition by an element of $GA_n(\mathbb{Z}) = GL_n(\mathbb{Z}) \rtimes \mathbb{R}^n$. A general moment map of $\mathbf{f}$ is of *toric type*.

Given two toric integrable systems (M_1, ω_1, F_1) and (M_2, ω_2, F_2), an isomorphism of toric integrable systems is a symplectomorphism

$$\varphi : (M_1, \omega_1) \to (M_2, \omega_2) \text{ such that } \varphi^* F_1 = F_2.$$

A toric (sub)-integrable system has characteristic properties for its moment map. This comes from the fact that a critical point is a fixed point of a sub-torus of $\mathbb{T}^k$, and near it the moment map can be linearized. Remembering the linear action of the torus given in Example 1.3.9, we have that a non-degenerated integrable system is of toric type if all its critical points have only elliptic or transverse components.

We give here some examples of toric systems, along with their image of moment map. Credits for these examples are going to Michèle Audin and her numerous books (see, among many others, [Aud96], [Aud04], [Aud08] and reference therein).

Example 1.4.2 ($S^1 \circlearrowleft \mathbb{C}^n$)**.** *If we assume $u = e^{i\theta} \in S^1$, we have the circle S^1 acting on $\mathbb{C}^n$ by $u\cdot(z_1, \dots, z_n) = (uz_1, \dots, uz_n)$. The associated vector field is*

$$\frac{\partial}{\partial \theta} \in TS^1 \text{ , that is: } X_H = \sum_{i=1}^{n} -y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}.$$

Example 1.4.3. *With the example above, it is easy to see that*

$$\mathbb{T}^n \circlearrowleft \mathbb{C}^n : (t_1, \dots, t_n)\cdot(z_1, \dots, z_n) = (t_1 z_1, \dots, t_n z_n)$$

is Hamiltonian, with moment maps $F(z_1, \dots, z_n) = \frac{1}{2}(|z_1|^2, \dots, |z_n|^2)$.

Example 1.4.4 (**The projective space**)**.** *We take $k_1, \dots, k_{n+1}$ relatively prime positive integers. In the symplectic reduction by the S^1-action on $\mathbb{C}^{n+1}$ (see Example 1.4.2)*

$$u\cdot(z_1, \dots, z_{n+1}) = (u^{k_1} z_1, \dots, u^{k_{n+1}} z_{n+1})$$

we have non-trivial isotropy groups: they are compact symplectic orbifolds called weighted projective space*. One can check they are actually projective spaces.*

Next, with Example 1.4.3, we can see that the action of $\mathbb{T}^{n+1}$ induced on the projective space is Hamiltonian.

A particular case of these orbifolds is the projective space $\mathbb{CP}^n$ with its standard symplectic form. We have $\mathbb{T}^n \circlearrowleft \mathbb{CP}^n$

$$(t_0, \dots, t_n)\cdot[z_0 : \dots : z_n] = [t_0 z_0, \dots, t_n z_n].$$

It is Hamiltonian, and the image of the moment map is the $(n+1)$-simplex.

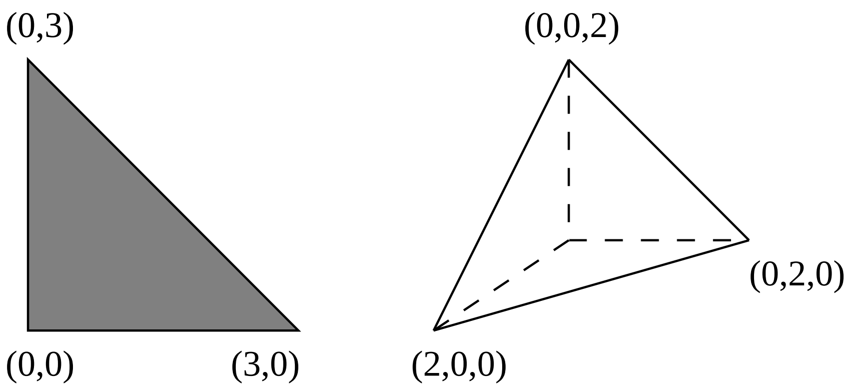

Figure 1.4: Toric systems of 2 and 3 degrees of freedom

Remark 1.4.5. *In the rest of the literature the term toric is generally reserved to the completely integrable case, but here we will also use it in the sub-integrable case. For instance, in [KT11], Karshon and Tolman use the expression* Hamiltonian $\mathbb{T}^k$-manifold *to designate a symplectic manifold with an (effective) Hamiltonian $\mathbb{T}^k$-action on it. Note also that the sub-integrable toric systems we will consider in this thesis will always be actual subsystems of integrable, but non-toric systems. This is the ransom of enlarging our study to semi-toric integrable systems: they can be seen as toric sub-integrable systems, or as non-toric integrable systems !*

1.4.1 Classification by moment polytopes

There is another issue of terminology over the term polytope, whose definition varies from one author to another. A polytope may be defined as a subset of $\mathbb{R}^k$ that admits a simplicial decomposition, but since we will only consider convex polytopes here we'll have no use of such a broad definition

Definition 1.4.6. *A (convex) polytope Δ of the affine space $\mathbb{R}^k$ is a finite intersection of m half-spaces $H_i^- = \{x \in \mathbb{R}^k \mid h_i(x) \leqslant b\}$ where $h_i = (h_{ij}) \in \mathbb{R}^{1\times n}$ is a linear form and $b \in \mathbb{R}^{m\times 1}$ the vector of constraints. The dimension of Δ is the unique d such that Δ is homeomorphic to a closed ball $\mathcal{B}^d(\mathbb{R}^k)$.*

An H-representation (with $H := (h_{ij}) \in \mathcal{M}_{m\times n}(\mathbb{R})$ for *Hyperplane*) of Δ is given by a minimal set of half-spaces. Such a representation is unique for full-dimensional polytopes.

For a polytope Δ of dimension d, a $(d-1)$-face, or *facet* of Δ, is the intersection of Δ and a bounding hyperplane $H_i = \{h_i^{-1}(b_i)\}$ of its H-representation. As a $(d-1)$-face is again a convex polytope (of dimension $d-1$), it admits a H-representation so by a descending recurrence we can define a r-face by being a facet of a $(r+1)$-face. The k-face is defined to be Δ. The 0-faces are called "vertices", the 1-faces are called "edges".

A compact convex polytope can also be defined as the convex hull of a finite set of points. A minimal set of points is given by the set of vertices, and it is unique.

Definition 1.4.7. *The Hasse diagram of a convex polytope Δ is an oriented graph whose set of nodes is $P(\Delta)$ the set of all r-faces, $0 \leqslant r \leqslant k-1$, and set of edges is given by the partial order relation of "frontier inclusion" introduced in Definition 1.3.14.*

The poset $(P(\Delta), \leq)$ is the second poset we encounter after $(\mathcal{W}_0^n(F), \preccurlyeq)$. We shall examine in Chapter 4 the relation between them.

Definition 1.4.8. *A polytope Δ is said to be* rational *if there exists $p \in \mathbb{N}$ such that $p \times H \in \mathcal{M}_{m\times n}(\mathbb{Z})$. It is said to be* normal *if for each vertex of Δ, the family of edges linked to that vertex is a basis of $\mathbb{R}^k$.*

We have a quite spectacular result concerning the image of the moment map:

Theorem 1.4.9 (Atiyah – Guillemin & Sternberg)**.** *Let $F : M^{2n} \to \mathbb{R}^k$ be a k-toric sub-integrable system. We have the following statements :*

1. *The fibers of F are connected.*
2. *$F(M)$ is a rational convex polytope called the* moment polytope*. It is the convex hull of the fixed points of F.*

There exists today many proofs of that theorem. The first proofs are due to Atiyah in [Ati82], and independently by Guillemin and Sternberg in [GS82]. Each of them used a different appproach, and even if the proof given by Atiyah in [Ati82] had non trivial gaps (which have been fixed since then, see [KT01] for instance), it is definitely more in the spirit of our treatment of the subject.

In the language of Hamiltonian group actions, Theorem 1.4.9 applies for compact connected Abelian Lie group actions. It admits a non-Abelian version, due to Kirwan:

Theorem 1.4.10 (Kirwan, [Kir84])**.** *If $G \circlearrowleft M$ is a Hamiltonian action, with G a connected compact Lie group and M a compact symplectic manifold, then*

- *The fibers of the associated moment map J are connected,*
- *The positive Weyl chamber of $\mathfrak{g}_+^*$ intersects $J(M)$ on a convex polytope.*

Remark 1.4.11. *Note that Atiyah – Guillemin & Sternberg theorem applies also in the sub-integrable case, and that the induced action need not be effective. We will indeed use the theorem in this context in Chapter 4.*

The theory of toric varieties is at the intersection of many domains. Symplectic geometry is one of them, but a toric manifold, or a toric variety, can be seen as the prototype of an algebraic variety that can be completely described by objects of combinatorial nature: the polytopes. Indeed, in the integrable toric case, the moment polytope captures all the information of the integrable system. This is how Delzant managed to provide a complete classification theorem for toric integrable systems

Theorem 1.4.12 (Delzant,[Del88],[Del90])**.** *Let*

$$(M_1^{2n_1}, \omega_1, F_1) \text{ and } (M_2^{2n_2}, \omega_2, F_2)$$

be two toric integrable systems such that the induced torus actions are effective. If $F_1(M_1) = F_2(M_2)$, then there exists a symplectomorphism $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ which is equivariant with respect to the torus actions and for which the following diagram commutes:

$$\begin{tikzcd} M_1 \arrow[r, "\varphi"] \arrow[d, "J_1"'] & M_2 \arrow[d, "J_2"] \\ J_1(M_1) \arrow[r, "id"] & J_2(M_2) \end{tikzcd}$$

Moreover, given a normal, convex, rational polytope Δ, there exists an integrable Hamiltonian system $(M_\Delta, \omega_\Delta, F_\Delta)$ such that $F_\Delta(M_\Delta) = \Delta$. Such an integrable system is unique up to equivariant symplectomorphism.

Delzant theorem is a classification theorem, but it also gives an explicit construction of an effective toric integrable system given a suiting polytope.

If we use the language of categories, it means that the category of all toric integrable systems is equivalent to the trivial category of all normal, rational, convex polytopes (such polytopes are called *Delzant polytopes*). Note also that if we quotient the category of Delzant polytopes by the natural action of $GA_n(\mathbb{Z}) = GL_n(\mathbb{Z}) \rtimes \mathbb{R}^n$, it is in bijection with the category of toric Abelian algebras $\mathbf{f}$.

Such classification has been extended to the orbifold case (see [LT97]). The efforts of many authors (Knop, Van Steirteghem, Woodward, Losev etc.) have resulted recently in a classification of integrable Hamiltonian action of compact, non-Abelian Lie groups.

1.5 Towards semi-toric integrable systems

The results above are a powerful theory for integrable systems yielding Hamiltonian actions of compact Lie groups, but there are several simple examples of mechanical systems where singularities that are not only elliptic occur. We are in particular interested in systems that have at least one unstable equilibrium point with a S^1-symmetry, but with these two features intermingled in a peculiar way. We will see that it is characteristic of focus-focus components of the critical points. We cite the following examples that exhibit non-elliptic critical points with the associated references for thorough investigation. The images for *Jaynes-Cumming-Gaudin* and the *Champagne bottle* models as well as the pinched torus are to be found at the pages 57 to 60:

Example 1.5.1 (Jaynes-Cumming-Gaudin model). *This model of $n-1$ spins coupled with an oscillator provides an example of a non-compact, but proper, semi-toric system of dimension $2n$. It has been extensively studied by the physicists (see [BD12] for instance) using techniques and language of algebraic geometers (the Jacobian, spectral curve, Bethe Ansatz), but the symplectic and semiclassical study of the $n = 1$ spin coupled with an oscillator was made in [PVN12], using the techniques we try to extend here to the arbitrary dimension.*

The model for $n = 1$ is the following: on the symplectic manifold $(S^2 \times \mathbb{R}^2, d\theta \wedge dz \oplus du \wedge dv)$, for the functions $H = ux + vy$ and $J = \dfrac{u^2 + v^2}{2} + z$ we have that $F = (H, J)$ is an integrable system. Here J is the momentum map for the Hamiltonian circle action of S^1 that corresponds to the simultaneous rotation around the z-axis of the sphere $\mathbb{S}^2$ and around the origin on $\mathbb{R}^2$.

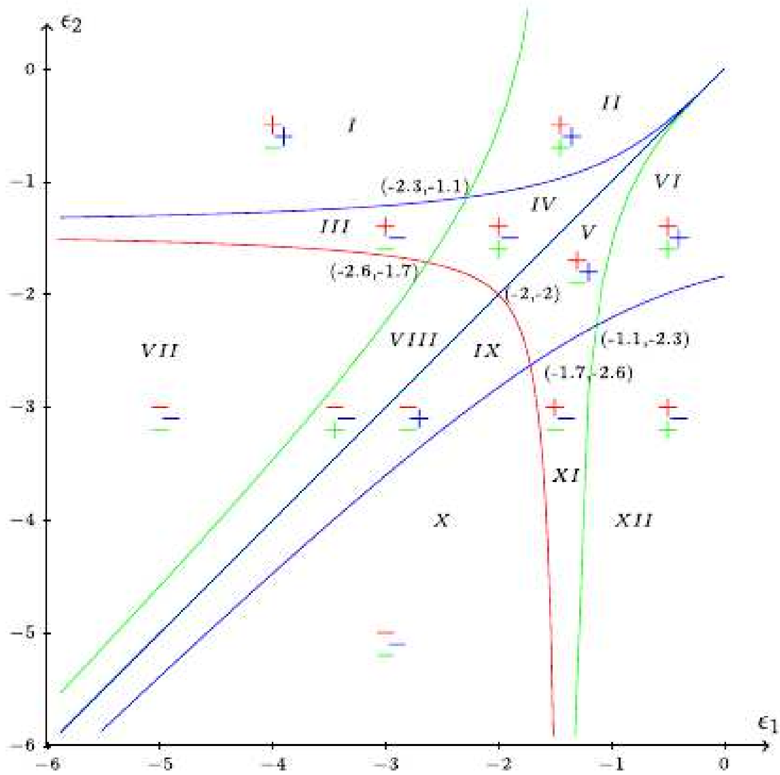

Fig. 8. The domains indicating the signs of the various discriminants, for three critical points of the system with two spins. Curves correspond to the vanishing of one discriminant. The color of these curves and of signs inside the domains label the critical points according to the following code: red stands for up–up, green for up–down, and blue for down–up. ($s = 1$). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$(\uparrow, \uparrow)$: one real root (unstable)
$(\uparrow, \downarrow)$: three real roots (stable)
$(\downarrow, \uparrow)$: one real root (unstable)
$(\downarrow, \downarrow)$: three real roots (always stable).

Figure 1.5: Spectral curves of the $n = 2$ Jaynes-Cumming-Gaudin model with the different regions represented

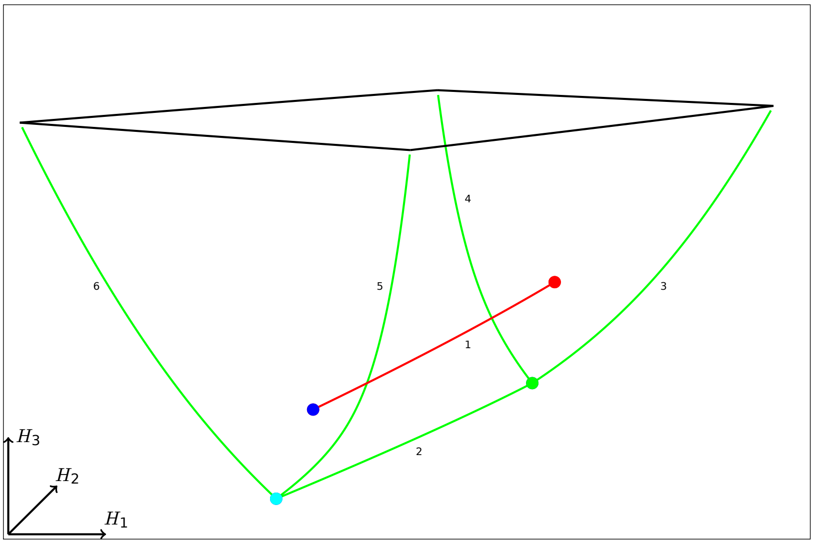

Figure 1.6: Credits goes to O.Babelon. Image of the Moment Map and bifurcation diagram for a point in region I. It has four faces, plus the one on top. E^3-points are vertices of the IMM. The $FF-E$ unstable points are on the front and the back faces. The line that joints the two $FF-E$ points is a path of $FF-X$ unstable points

The singularities of the coupled spin-oscillator are non-degenerate and of elliptic-elliptic, transversally-elliptic or focus-focus type. It has exactly one focus-focus singularity at the "North Pole" $((0,0,1),(0,0)) \in S^2 \times \mathbb{R}^2$ *and one elliptic-elliptic singularity at the "South Pole"* $((0,0,-1),(0,0)) \in S^2 \times \mathbb{R}^2$*. Now depending on the values of the different coupling, the system may or may not enter in our framework. We reproduce below the possible values of the coupling for the 6-dimensional case corresponding to the model with* 2 *spins coupled with an oscillator, and we indicate the ones for which we have a suitable system and then give its bifurcation diagram. We give the pictures along with the captions.*

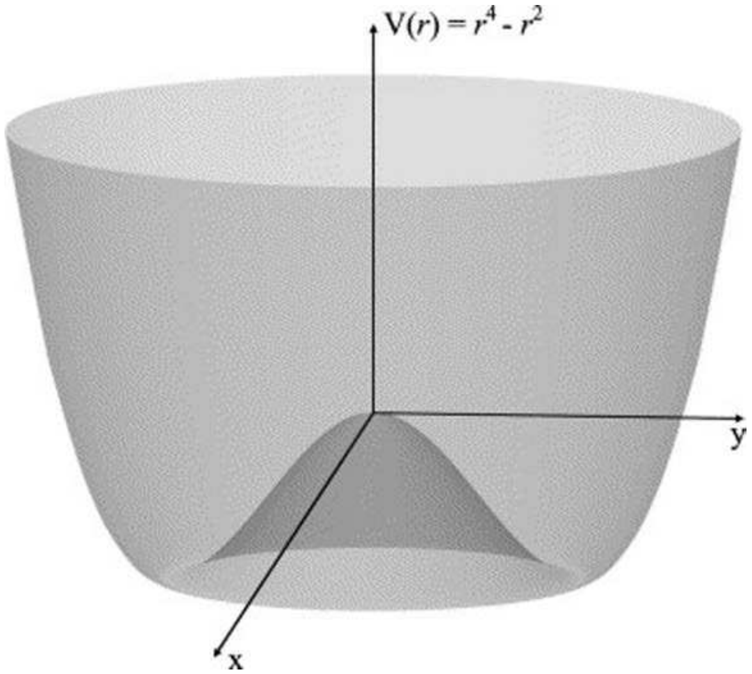

Figure 1.7: Champagne bottle potential function $V(r)$ with $r^2 = x^2 + y^2$. The origin of the coordinate system is at the critical point on top of the potential hump. The space below the critical point has the surface topology $\mathbb{S}^2 \times S^1$ and above $\mathbb{S}^3$. Therefore no set of global action-angle variables can exist.

Example 1.5.2 (Champagne bottle, [Bat91])**.** *It modelizes a punctual mass moving on a plane, under a double-well potential with rotational invariance around the origin: $V(x, y) = -(x^2 + y^2) + (x^2 + y^2)^2$. The symplectic manifold is $T^*\mathbb{R}^2 = \mathbb{R}^4_{(x,y,p_x,p_y)} = \mathbb{R}^4_{(r,\theta,p_r,p_\theta)}$. The Hamiltonian accounting for the energy conservation is*

$$H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + -(x^2 + y^2) + (x^2 + y^2)^2$$
$$H(r, \theta, p_r, p_\theta) = \underbrace{\frac{1}{2}(p_r^2 + \frac{1}{r^2}p_\theta^2)}_{\text{kinetic energy}} + \underbrace{r^4 - r^2}_{\text{potential energy}}.$$

Since the system is invariant under rotation, the other conserved quantity is simply $I = p_\theta$. The moment map is $F = (H, I)$. We have:

$$dH = p_x dp_x + p_y dp_y - 2xdx - 2ydy + 4x^3dx + 4xy^2dx + 4x^2ydy + 4y^3dy,$$
$$dI = p_x dy - p_y dx.$$

So in 0 we have that $dH \wedge dI = 0$ if and only if $p_x = p_y = 0$: $(0, 0, 0, 0)$ is a critical point. It is actually a fixed point.

When $r \neq 0$, we have

$$\begin{aligned} dH &= p_r dp_r + \frac{1}{r^2} dp_\theta + (\frac{p_\theta}{r^3} - 2r + 4r^3) dr, \\ dI &= dp_\theta. \end{aligned}$$

Note that this example, as well as the next one, is not, strictly speaking a semi-toric integrable system, since the moment map is not proper.

Example 1.5.3 (Spherical pendulum)**.** *This semi-toric system is probably the most ancient integrable system treated in literature that exhibited semi-toric behaviour. We can go back to the XVIIth century with Huygens for a detailed study of this simple but fundamental example.*

Here, the symplectic manifold will be the cotangent space of the sphere of radius 1: $(T^*S^2, \omega_{T^*S^2}) = \{(u,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|u\| = 1 \ , \ \langle u, v\rangle\}$. *The Hamiltonian accounting for the energy conservation is*

$$H(\underbrace{\varphi, \theta}_{sphere}, \underbrace{\xi, \eta}_{fiber}) = \frac{1}{2}\|u\| + z.$$

The system is invariant by rotation around the z-axis. Thus the projection of the angular momentum on the z-axis is a preserved quantity by the evolution of the system:

$$I = \langle v \wedge u, e_z\rangle.$$

This is the second component of the moment map. Its Hamiltonian flow is the rotation around the z-axis. The critical points of $F = (H, I)$ are the following

- *An elliptic-elliptic fixed point at $(u, v) = (-1, 0)$: the pendulum is at the south pole and with zero velocity.*
- *A focus-focus fixed point at $(u, v) = (1, 0)$: the pendulum is at the north pole and with zero velocity.*
- *Critical points of rank 1:* $\mathrm{rk}(dF(u, v)) = 1$ *if and only if*

  $$z + \lambda^2 = 0 \text{ and } v = \lambda^{-1} u \wedge e_z \text{ for } \lambda \neq 0$$

  This is a 1-parameter family of transversally elliptic critical points. They are "particular trajectories": a family of horizontal circles parametrized by their latitude, covered with constant speed depending of the circle.

South pole is a stable point for H and for I. North pole is an unstable point for H, but not for I. As we already mentionned, this example, as simple as it seems, doesn't fit in our framework, as its moment map is not proper. However, in the article [PRVN11] Pelayo, Ratiu and San Vũ Ngọc provided tools to deal with such systems in dimension $2n = 4$. In particular, they give a full study of this example (it was actually one of the motivation of the article).

The results given in this article are formulated by making specific assumptions of the map $\check{F}$ introduced in Section 1.2.1. With these assumptions, the authors are able to recover the connectedness of the fibers and to describe the image of the moment map.

Example 1.5.4 (Lagrange top and Kowalewskaya top,[CK85])**.** *These two examples of spining tops are non-degenerate integrable systems that contains focus-focus critical points, but they also contains hyperbolic points. As a result, we cannot apply our results on these systems. This shows that the class of systems we are treating is far from covering the family of non-degenerate integrable systems. Integrating the study of the image of the moment map in the hyperbolic case would be a major improvement of this theory.*

Example 1.5.5 (Quantum chemistry)**.** *Physicists and chemists were the first to become interested in semi-toric systems. Semi-toric systems appear naturally in the context of quantum chemistry. Many groups have been working on this topic, trying to understand the influence of features of a classical system on the spectrum of its quantized counterpart. We cite here the work of physicists and chemists that have exhibited quantum molecular systems whose classical counterparts integrable systems with focus-focus singularities: the CO_2 molecule in [CDG⁺04], the HCP molecule in [JGS98] and the Li_2C_2N molecule in [JST03]. One of the motivation of our work was the demand of these researchers to produce, in the spirit of semi-classical analysis, a classification of such systems by invariants that could be recovered from the joint quantum spectrum.*

We chose examples that all have in common to exhibit focus-focus critical points. The systems we will focus on are the ones where we authorize critical points with elliptic and focus-focus components. The natural question is to ask to what extent the structure results of Atiyah, Guillemin & Sternberg and the classification result of Delzant through moment polytopes can be generalized to these systems. That question is actually part of a greater research project initiated by A. Pelayo, T. S. Ratiu and San Vũ Ngọc. It is described in [PVN11b] and its long-term aim is to give a classification of integrable systems with "concrete" objects. Although it is an important goal in itself, the original motivation was coming from quantum physics: the hope is also to be able to detect the trace of these objects on their quantized counterpart through the use of semi-classical analysis.

1.5.1 Definition of c-almost-toric systems

We follow in this subsection the path given in the paper [VN07], and give precisions over the terminology introduced first by Symington in [Sym01]. First, a notation: $\check{F}^i = (f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n)$, $\check{F}^{\leqslant r} = (f_{r+1}, \dots, f_n)$. When i or r is equal to 1, we just note $\check{F}$.

Definition 1.5.6. *A proper integrable system F is called* almost-toric *if all its singularities are non-degenerate and without hyperbolic blocks: $k_{\mathrm{h}} = 0$. We say in particular that it is r*-almost-toric*, and we write $f \in AT_r(M)$ if $\check{F}^{\leqslant r}$ is a moment map for a (global) Hamiltonian $\mathbb{T}^{n-r}$-action.*

We already saw that F generates a strongly Hamiltonian action on M of an Abelian Lie group $U \simeq \mathbb{R}^c \times \mathbb{T}^{n-c}$. The torus $\mathbb{T}^{n-c}$ is the "maximal" torus action generated by F. Thus, the number c is well defined: it is called the complexity *of the almost-toric system.*

We have that $AT_r(M)$ is the set of almost-toric systems of complexity less than or equal to r, and we have that $AT_r(M) \subseteq AT_{r'}(M)$ if $r \leqslant r'$. The set $AT_1(M)$ is the set of **semi-toric** *systems.*

Note that with our choice of definition of the torus $\mathbb{T}^r = (S^1)^r$ and $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, $F \in AT_r(M)$ if and only if the $n-r$ last components of F are 2π-periodic.

This definition coincides in the case of integrable systems with the definition of complexity given by Karshon and Tolman in [KT11]. They define the complexity of a symplectic manifold M^{2n} with a Hamiltonian $\mathbf{T}$-action (a Hamiltonian $\mathbf{T}$-manifold) as half the dimension of the symplectic quotient $M/\!\!/\mathbf{T}$ at a regular point. Karshon in [Kar99], gives a classification of 4-dimensional Hamiltonian S^1-manifold (Hamiltonian $\mathbf{T}$-manifold of complexity 1). Karshon and Tolman in [KT01], [KT03] and [KT11] extended this classification to arbitrary dimensions, provided that the quotient $M/\!\!/\mathbf{T}$ is always a two-dimensional topological manifold (what they name "tallness" condition).

This amounts to consider all $(n-1)$-toric sub-integrable systems with our terminology but without taking advantage of the particular nature of our systems: the fact that the torus action is coming from an integrable system, and the properties of focus-focus singularities. While in our case complexity is due to a lack of periodic Hamiltonian flows, in their cases systems may not even have enough Hamiltonian flows at all.

We could have treated semi-toric systems as Hamiltonian $\mathbf{T}$-manifold of complexity 1 with an additional function f_1 commuting with $\check{F}$, but this seemed unnatural to us, given how we cannot treat separately f_1 and $\check{F}$ in a focus-focus singularity. However understanding our results in the approach of Karshon and Tolman would surely be an interesting work to realize.

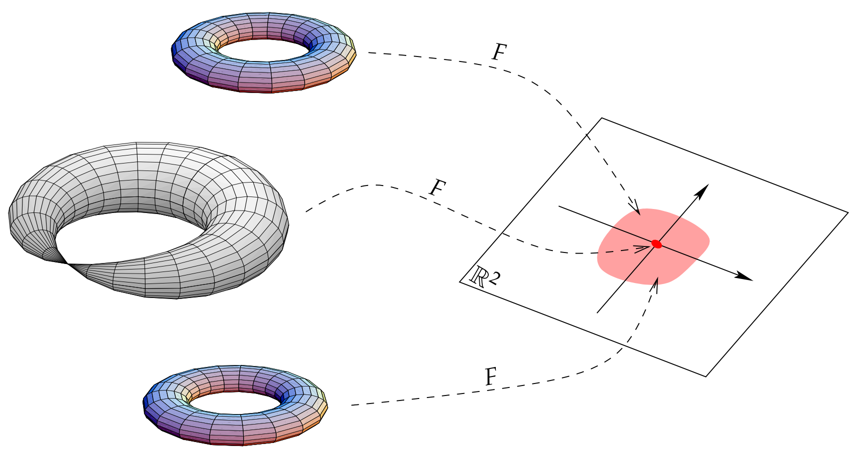

Figure 1.8: Local model of singular fibration in dimension 2 near a focus-focus point.

We can, as in the toric case, define integrable systems of almost-toric type by saying that F is of almost-toric type of complexity c if there is a local diffeomorphism Ψ of $F(M)$ to its image such that $\Psi \circ F$ is almost-toric of complexity c.

Remark 1.5.7. *Here we chose the last components of F to yield the torus action, but it is of course only a matter of convention. The relevant thing is the maximal dimension of the global Hamiltonian torus action. A consequence of our definition is that an almost toric system of complexity c is c-almost-toric only up to a permutation of the components.*

We chose to fix the components in the definition to simplify the statements.

One can say that c-almost-toric almost have a $\mathbb{T}^k$-action as we have a good knowledge of non-toric Hamiltonians, whose dynamic live on pinched tori (the level sets of focus-focus singularities). Here, in the semi-toric case, we necessarily have that it is always the same component that fails to have a periodic flow.

We give below a $3D$-picture of a pinched torus. It represents the non-degenerate critical level sets of points that have the same image by the moment map as the focus-focus critical point. Of course, this picture is only here to help the people that are dealing with integrable systems to get a clearer representation of the dynamics near focus-focus singularity, with the "pseudo-hyperbolic" flow and "pseudo-elliptic" flow intermingled, since focus-focus leaves exist only in manifolds of dimension $\geqslant 4$.

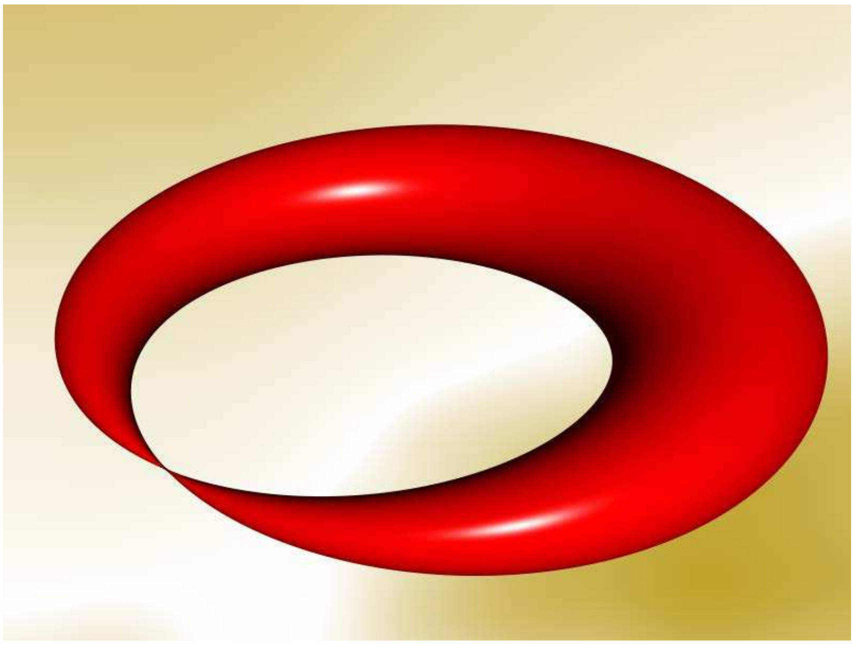

Figure 1.9: Pinched torus, leaf of a focus-focus critical point. The pinched cycle is the critical point, the other points are regular points on a critical leaf.

Remark 1.5.8. *For the sake of clarity, we will use in the semi-toric case the obvious notations* $FF^{k_\mathrm{f}} - E^{k_\mathrm{e}} - X^{k_\mathrm{x}}$ *when dealing with concrete example, instead of giving the quadruplet* $\Bbbk$ *(let's note that in the almost-toric case we only deal with critical points with* $k_\mathrm{h} = 0$*). However the notation* $\Bbbk$ *is helpful for general statements.*

The great difference between toric and almost-toric (and semi-toric) cases lies in the fact that at singular points with focus-focus component, not only the torus action ceases to be effective, but the torus action *itself* is destroyed. In 3 dimensions, almost-toric actions we consider here are Hamiltonian actions of $\mathbb{T}^2 \times \mathbb{R}$, a non-compact Lie group. Among them, the semi-toric case is in a certain way the simplest relaxation of the toric assumption: considered from the point of view of algebraic geometry, semi-toric manifolds are Jacobians of singular algebraic curves. To this, we add to this the integrability of the underlying Hamiltonian system.

Assumption 1.5.9. *From now on, we will only consider integrable system on* **compact** *manifolds to simplify proofs.*

In [PRVN11] the authors consider integrable systems on non-compact manifolds but with proper moment maps. The conclusions concerning the description of the image of the moment map by a family of polytopes are almost the same as in the compact case, provided the image verifies a condition of "non-vertical tangency". The treatment of the case of a non-proper moment map on a non-compact manifold is still possible, but requires assumptions we'd wish to avoid, such as connectedness of the fibers.

1.5.2 Isomorphisms of semi-toric systems

As the long term objective is to get a classification of semi-toric systems, we fix here an equivalence relation on semi-toric systems.

Definition 1.5.10. *An isomorphism between two semi-toric integrable systems*

$$(M_1, \omega_1, F_1) \text{ and } (M_2, \omega_2, F_2)$$

is a pair (φ, V), where $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ is a symplectomorphism and V is a local diffeomorphism of $\mathbb{R}^n$ such that $\varphi^* F_1 = V \circ F_2$ and $\varphi^* \check{F}_1 = \check{F}_2$.

If there exists such an isomorphism between two semi-toric integrable systems, we say that they are ST-equivalent and we note $(M_1, \omega_1, F_1) \sim_{ST} (M_2, \omega_2, F_2)$.

A V like in the definition is of the form

$$V(c_1, \ldots, c_n) = (V_1(c_1, \ldots, c_n), c_2, \ldots, c_n) \text{ with } \frac{\partial V_1}{\partial c_1} \neq 0.$$

An isomorphism of semi-toric integrable systems is a symplectomorphism that preserves the semi-toric foliation, and also the toric part of the moment map.

1.5.3 Structure and classification results of semi-toric integrable systems

Results concerning the connectedness of the fibers, the retrieval of convex polytopes from the image of the moment map, and how we can use them to classify and construct semi-toric integrable systems have been obtained in dimension $2n = 4$ by San Vũ Ngọc and Álvaro Pelayo ([VN07], [PVN09], [PVN11a]). The new thing with semi-toric systems is that one has to "cut" the image of the moment map to get a polytope, and as this "cut" is not unique, we have several polytopes. The existence of these results was the main motivation of this thesis.

Let $F = (J, H) : M^4 \to \mathbb{R}^2$ be a semi-toric integrable system, and let $D = F(M) \subseteq \mathbb{R}^2$ be the image of the moment map, D_r the set of regular values. We suppose that J is a proper moment map for a S^1-action. Let $\{c_i = (x_i, y_i), i = 1, \ldots, m_f\} \subseteq \mathbb{R}^2$ be the set of focus-focus critical points. The fact that it is a finite set of points is easy to see from Eliasson normal form (see [VN07] or Chapter 3). We order the points by increasing abscissa, and make the assumption that each focus-focus critical point has different abscissa (we say the semi-toric system is *simple*).

For each i and for some $\epsilon \in \{-1, +1\}$ we define $\mathcal{L}^i_\epsilon$ to be the vertical half-line starting at c_i and going to $+\infty$ or $-\infty$ respectively: $\mathcal{L}^i_\epsilon = c_i + \epsilon \cdot \mathbb{R}_+ \cdot \vec{e_y}$. Given $\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_{m_f}) \in \{-1, +1\}^{m_f}$, we define the line segment $l^i_\epsilon = D \cap \mathcal{L}^i_\epsilon$ and

$$\mathbf{l}^{\vec{\epsilon}} = \bigcup_{i=1}^{m_f} l^i_{\epsilon_i}.$$

Let $\mathcal{A}^2_{\mathbb{Z}}$ be the plane $\mathbb{R}^2$ equipped with its standard integral affine structure (an integral affine structure is an atlas of charts with transition maps in $GA_2(\mathbb{Z})$, see Chapter 5 for more details on integral affine structures). The group of automorphisms of $\mathcal{A}^2_{\mathbb{Z}}$ is the integral affine group $GA_2(\mathbb{Z})$. We denote by $\mathcal{T}$ the subgroup of $GL(2, \mathbb{Z}) \ltimes \mathbb{R}^2$ which leaves a vertical line (with orientation) invariant. In other words an element of $\mathcal{T}$ is a composition of a vertical translation and a power of the matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Theorem 1.5.11 (San Vũ Ngọc, [VN07]). *Given any $\epsilon \in \{-1, +1\}^{m_f}$, we have that the fibers of F are connected, and there exists a homeomorphism $v : D \to v(D)$ such that:*

1. *$v|_{D \setminus l^\epsilon}$ is an affine diffeomorphism onto its image.*
2. *v preserves J: $v \circ F = (J, v_2(J, H))$.*
3. *For $i = 1, \ldots, m_f$, there is an open ball B around $c \in \mathring{l}_i$ such that $v|_{D \setminus l^\epsilon}$ has a smooth extension on each domain $\{(x, y) \in D_r,\ x \leqslant x_i\}$ and $\{(x, y) \in D_r,\ x \geqslant x_i\}$ and we have the formula*

   $$\lim_{\substack{(x,y)\to c \\ x<x_i}} df_{(x,y)} = \lim_{\substack{(x,y)\to c \\ x>x_i}} T^{k(c_i)} df_{(x,y)}$$

   where $k(c_i)$ is the multiplicity of c_i.
4. *The image Δ of v is a rational convex polygon. Such a v is unique modulo a left composition by a transformation in $\mathcal{T}$.*

As a result we have that the image of the moment map can be cut and then straightened affinely to a polytope. Hence, it can be seen as a generalization of Atiyah – Guillemin & Sternberg Theorem 1.4.9 to the case of semi-toric integrable systems. The cuts, and hence the polytopes are not unique: these generalized Delzant polygones come in a family on which $(\mathbb{Z}/(2\mathbb{Z}))^{m_f}$ acts freely and transitively.

Remark 1.5.12. *In the next chapter, we make the assumption to consider systems whose focus-focus singularities are reduced to a single pinched torus. Thus for us, $k(c_i)$ will always be equal to 1. The situation can be more complicated if we consider semi-toric systems whose focus-focus singularities have several pinches.*

Theorem 1.5.13 (Álvaro Pelayo & San Vũ Ngọc, [PVN09], [PVN11a]). *On (M^4, ω), to each semi-toric integrable system we can associate the following list L of invariants of various nature:*

- *The number $0 \leqslant m_f < \infty$ of focus-focus points.*
- *The rational weighted polygone invariant given by Theorem 1.5.11: it is the orbit*
$$\left((\mathbb{Z}/(2\mathbb{Z}))^{m_f} \times \{T^k \mid k \in \mathbb{Z}\}\right) \star \left(\Delta, \mathbf{l}^{\vec{\epsilon}}, \vec{\epsilon}\right)$$
where Δ is a rational weighted convex polygone, $\mathbf{l}^{\vec{\epsilon}}$ a family of vertical segments and $\vec{\epsilon}$ a m_f-tuple of ± 1's. The action $\star$ is defined in the article we gave as references.
- *For each focus-focus critical leaf $F^{-1}(c_i)$, the symplectic invariant S_i^∞ defined in Section 3.3: it is a Taylor serie in two variables.*
- *The volume invariant: for each focus-focus critical value c_i, the quantity*
$$h_i = \frac{1}{n!} \int_{J^{-1}(c_i) \cap H^{-1}(]-\infty, c_i])} \omega^n.$$
- *The twisting index: a geometrical invariant associated to the choice of local normal form near each focus-focus point.*

These invariants are characteristic of semi-toric integrable systems in dimension 4: two systems $(M^4, \omega, (J_1, H_1))$ and $(M^4, \omega, (J_2, H_2))$ are isomorphic in the sense of Definition 1.5.10 if and only if they have the same list of invariants.

Moreover, given such a list of invariants L, there exists a unique (up to isomorphism) semi-toric system $(M_L^4, \omega_L, (J_L, H_L))$ whose list of invariants is L.

This theorem generalizes Delzant's classification Theorem 1.4.12 to semi-toric integrable systems in dimension $2n = 4$. Our primary goal was, as we already mentionned, to extend these results to any dimension.

Chapter 2

Eliasson normal form of focus-focus-elliptic singularitites

In this chapter we reproduce an article that appeared in the *Acta Mathematica Vietnamica* issue of February. This was the first task we achieved during our thesis. In it, we give the complete, detailed proof of a normal form theorem first stated by Eliasson in 1986 [Eli84] in the focus-focus case of dimension 4.

2.1 Introduction and exposition of the result

In his PhD Thesis [Eli84], Eliasson proved some very important results about symplectic linearisation of completely integrable systems near non-degenerate singularities, in the $\mathcal{C}^\infty$ category. However, at that time the so-called elliptic singularities were considered the most important case, and the case of focus-focus singularities was never published. It turned out that focus-focus singularities became crucially important in the last 15 years in the topological, symplectic, and even quantum study of Liouville integrable systems. In this article we prove that a near a focus-focus singularity, smooth integrable systems with two degrees of freedom are symplectically equivalent to their quadratic normal form.

The so-called *focus-focus* model is the integrable system (q_1, q_2) on $\mathbb{R}^4 = T^*\mathbb{R}^2$ equipped with the canonical symplectic form $\omega_0 := d\xi_1 \wedge dx_1 + d\xi_2 \wedge dx_2$ given by:

$$q_1 := x_1\xi_1 + x_2\xi_2 \quad \text{and} \quad q_2 := x_1\xi_2 - x_2\xi_1. \tag{2.1}$$

The focus-focus model will be denoted by $\mathcal{Q}_f := (\mathbb{R}^4, \omega_0, (q_1, q_2))$.

If $m \in M$ is a critical point for a function $f \in \mathcal{C}^\infty(M)$, we denote by $\mathcal{H}_m(f)$ the Hessian of f at m, which we view as a quadratic form on the tangent space $T_m M$.

Definition 2.1.1. *Let f_1, f_2 be $\mathcal{C}^\infty$ functions on a 4-dimensional symplectic manifold M, such that $\{f_1, f_2\} = 0$. Here the bracket $\{\cdot,\cdot\}$ is the Poisson bracket induced by the symplectic structure. A point $m \in M$ is a* ***focus-focus*** *critical point for the couple (f_1, f_2) if:*

- *$df_1(m) = df_2(m) = 0$;*
- *the linearized system $(T_m M, (\mathcal{H}_m(f_1), \mathcal{H}_m(f_2)))$ is linearly symplectomorphic to the linear model $\mathcal{Q}_f$: there exists a linear symplectomorphism $U : \mathbb{R}^4 \to T_m M$ and an invertible matrix $G \in \mathrm{GL}_2(\mathbb{R})$ such that*

$$\mathcal{H}_m(f_i) \circ U = G(q_1, q_2), \qquad \forall i = 1, 2.$$

This is the non-degeneracy condition as defined by Williamson [Wil36]. It is equivalent to fact that a generic linear combination of the Hamiltonian matrices corresponding to $\mathcal{H}_m(f_1)$ and $\mathcal{H}_m(f_2)$ has four distinct complex eigenvalues.

The purpose of this paper is to give a new proof of Theorem 1.2.23, which was stated in [Eli84]. We give here its formulation in the dimension 4 focus-focus case :

Theorem 2.1.2. *Let (M, ω) be a symplectic 4-manifold, and f_1, f_2 smooth functions on M with $\{f_1, f_2\} = 0$. Let m be a focus-focus critical point for (f_1, f_2). Let $F = (f_1, f_2) : M \to \mathbb{R}^2$.*

Then there exists a local symplectomorphism $\Psi : (\mathbb{R}^4, \omega_0) \to (M, \omega)$, defined near the origin, and sending the origin to the point m, and a local diffeomorphism $G : \mathbb{R}^2 \to \mathbb{R}^2$, defined near 0, and sending 0 to $F(m)$, such that

$$F \circ \Psi = G(q_1, q_2).$$

In the holomorphic category, this result was proved by Vey [Vey78]. A proof of the result in the smooth category is sketched in [Zun02]. The prepublished proof on ArXiv gave the idea to Marc Chaperon to a simpler proof he published (see [Cha]). Both proofs relie on non-trivial results such as Sternberg linearization theorem near hyperbolic point. Our proof takes a different path, but of course the obstacles to be overcomed are similar in all proofs.

The geometric content of our normal form theorem is that the foliation defined by F may, in suitable symplectic coordinates, be made equal to the foliation given by the *quadratic part* of F. With this in mind, the theorem can be viewed as a "symplectic Morse lemma for singular Lagrangian foliations".

The normal form $\tilde{G}$ and the normalization Ψ are not unique. However, the degrees of liberty are well understood. We cite the following results for the reader's interest, but they are not used any further in this chapter.

Theorem 2.1.3 ([VN03]). *If φ is a local symplectomorphism of $(\mathbb{R}^4, 0)$ preserving the focus-focus foliation $\{q := (q_1, q_2) = const\}$ near the origin, then there exists a unique germ of diffeomorphism $G : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$q \circ \varphi = G \circ q, \tag{2.2}$$

and G is of the form $G = (G_1, G_2)$, where $G_2(c_1, c_2) = \varepsilon_2 c_2$ and $G_1(c_1, c_2) - \varepsilon_1 c_1$ is flat at the origin, with $\varepsilon_j = \pm 1$.

This theorem is generalized in Chapter 3.

Theorem 2.1.4 ([MZ04]). *If φ is a local symplectomorphism of $(\mathbb{R}^4, 0)$ preserving the quadratic map (q_1, q_2), then φ is the composition $A \circ \chi$ where A is a linear symplectomorphism preserving (q_1, q_2) and χ is the time-1 Hamiltonian flow of a smooth function of (q_1, q_2).*

2.1.1 Complex variables

Since Theorem 2.1.2 is a local theorem, we can always invoke the Darboux theorem and formulate it in local canonical coordinates in $(\mathbb{R}^4, \omega_0)$. It is well known that focus-focus components are conveniently dealt with in complex coordinates, as follows.

We set $z_1 := x_1 + ix_2$ and $z_2 := \xi_1 + i\xi_2$. Then $q_1 + iq_2 = \overline{z}_1 z_2$, and the Hamiltonian flows of q_1 and q_2 is

$$\varphi^t_{q_1} : (z_1, z_2) \mapsto (e^t z_1, e^{-t} z_2) \text{ and } \varphi^s_{q_2} : (z_1, z_2) \mapsto (e^{is} z_1, e^{is} z_2). \tag{2.3}$$

Notice also that the Poisson bracket for real-valued functions f,g is :

$$\{f, g\} = 2\left(-\frac{\partial f}{\partial \bar{z}_1}\frac{\partial g}{\partial z_2} + \frac{\partial f}{\partial z_2}\frac{\partial g}{\partial \bar{z}_1} - \frac{\partial f}{\partial z_1}\frac{\partial g}{\partial \bar{z}_2} + \frac{\partial f}{\partial \bar{z}_2}\frac{\partial g}{\partial z_1}\right).$$

2.2 Birkhoff normal form for focus-focus singularities

In this section we recall why Theorem 2.1.2 holds in a formal context (i.e.: with formal series instead of functions), and use the formal result to solve the problem modulo a flat function. For people familiar with Birkhoff normal forms, we compute here simultaneously the Birkhoff normal forms of commuting Hamiltonians. We are not aware of this particular result being already published, but of course the ideas are standard, and we give them for the sake of completeness.

2.2.1 Formal series

We consider the space $\mathcal{E} := \mathbb{R}[\![x_1, x_2, \xi_1, \xi_2]\!]$ of formal series, with the usual convergence associated with the global degree in the variables. Thus, if $\mathring{f} \in \mathcal{E}$, we write

$$\mathring{f} = \sum_{N=0}^{+\infty} \mathring{f}^N \text{ , with } \mathring{f}^N = \sum_{\substack{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^n \\ |\alpha|+|\beta|=N}} \mathring{f}_{\alpha\beta}\, x^\alpha \xi^\beta \text{ and } \mathring{f}_{\alpha\beta} \in \mathbb{R}.$$

Given a smooth function $f \in \mathcal{C}^\infty(\mathbb{R}^4, 0)$, its Taylor expansion is a formal series in $\mathcal{E}$; we will denote it by $\mathcal{T}(f) \in \mathcal{E}$. It will be convenient to introduce the following definitions

Definition 2.2.1. $\mathring{\mathcal{O}}(N) := \{\mathring{f} \in \mathcal{E} \mid \quad \mathring{f}_{\alpha\beta} = 0, \ \forall\, |\alpha| + |\beta| < N\}$.

Definition 2.2.2. *A smooth function $f \in \mathcal{C}^\infty(\mathbb{R}^4, 0)$ belongs to $\mathcal{O}(N)$ if one of the 3 equivalent conditions is fulfilled:*

1. *f and all its derivatives of order $< N$ at 0 are 0.*
2. *There exists a constant $C_N > 0$ such that, in a neighbourhood of the origin,*
$$|f(x_1, x_2, \xi_1, \xi_2)| \leq C_N(\|x\|_2^2 + \|\xi\|_2^2)^{N/2}. \tag{2.4}$$
3. *$\mathcal{T}(f) \in \mathring{\mathcal{O}}(N)$.*

We use the notation $\|\cdot\|_2$ for the Euclidean norm in $\mathbb{R}^2$. The equivalence of the above conditions is a consequence of the Taylor expansion of f. Recall, however, that if f were not supposed to be smooth at the origin, then the estimates (2.4) alone would not be sufficient for implying the smoothness of f.

Lemma 2.2.3. *Let $f \in \mathcal{C}^\infty(\mathbb{R}^k; \mathbb{R}^k)$ and $g \in \mathcal{C}^\infty(\mathbb{R}^k; \mathbb{R})$. If $f(0) = 0$ and $g \in \mathcal{O}(N)$, then $g \circ f \in \mathcal{O}(N)$. Moreover if f and g depend on a parameter in such a way that their respective estimates (2.4) are uniform with respect to that parameter, then the corresponding $\mathcal{O}(N)$-estimates for $g \circ f$ are uniform as well.*

Proof. In view of the estimates (2.4), given any two neighbourhoods of the origin U and V, there exists, by assumption, some constants C_f and C_g such that

$$\|f(X)\|_2 \leq C_f \|X\|_2 \quad \text{ and } \quad |g(Y)| \leq C_g \|Y\|_2^N ,$$

for $X \in U$ and $Y \in V$. Since $f(0) = 0$, we may choose V such that $f(U) \subset V$. So we may write

$$|g(f(X))| \leq C_g \|f(X)\|_2^N \leq C_g C_f^N \|X\|_2^N ,$$

which proves the result. □

Definition 2.2.4. *A function $f \in \mathcal{C}^\infty(\mathbb{R}^k, 0)$ is **flat** at the origin, or $\mathcal{O}(\infty)$, if for all $N \in \mathbb{N}$, it is $\mathcal{O}(N)$. Its Taylor expansion is zero as a formal series.*

Of course, smooth functions can be flat and yet non-zero in a neighbourhood of 0. The most classical example is the function $x \mapsto \exp(-1/x^2)$. The following Borel lemma is standard:

Lemma 2.2.5 ([Bor95])**.** *Let $\mathring{f} \in \mathbb{R}[\![x_1, \dots, x_k]\!]$. Then there exists a function $\tilde{f} \in \mathcal{C}^\infty(\mathbb{R}^k)$ whose Taylor series is $\mathring{f}$.*

We define the Poisson bracket for formal series just like in the smooth setting: for $A, B \in \mathcal{E}$,

$$\{A, B\} = \sum_{i=1}^{2} \frac{\partial A}{\partial \xi_i} \frac{\partial B}{\partial x_i} - \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial \xi_i}.$$

The same notation will denote the smooth and the formal bracket, depending on the context. The Poisson bracket commutes with taking Taylor expansions: for formal series A, B,

$$\{\mathcal{T}(A), \mathcal{T}(B)\} = \mathcal{T}(\{A, B\}). \tag{2.5}$$

From this we deduce, if we let $\mathcal{D}_N$ be the subspace of homogeneous polynomials of degree N in the 4 variables (x, ξ):

$$\{\mathcal{O}(N), \mathcal{O}(M)\} \subset \mathcal{O}(N + M - 2) \text{ and } \{\mathcal{D}_N, \mathcal{D}_M\} \subset \mathcal{D}_{N+M-2}.$$

We also define $\mathrm{ad}_A : f \mapsto \{A, f\}$. The usual "exp −ad" commutative diagram for Lie groups is valid in this context:

Lemma 2.2.6. *For $f \in \mathcal{C}^\infty(\mathbb{R}^4; \mathbb{R})$ and $A \in \mathcal{O}(3)$ a smooth function, let φ_A^t be the Hamiltonian flow of A at time t. Then we have*

$$\mathcal{T}((\varphi_A^t)^* f) = \exp(t \, \mathrm{ad}_{\mathcal{T}(A)}) f,$$

for each $t \in \mathbb{R}$ for which the flow on the left-hand side is defined.

Notice that, since $\mathcal{T}(A) \in \mathring{\mathcal{O}}(3)$, the right-hand side

$$\exp(t \, \mathrm{ad}_{\mathcal{T}(A)}) f = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\mathrm{ad}_{\mathcal{T}(A)}\right)^k f$$

is always convergent in $\mathcal{E}$. This lemma can be easily proved by induction on the degree, with the help of Lemma 2.2.3. As a consequence, $\exp \mathrm{ad}_{\mathcal{T}(A)}$ is formally symplectic, which gives

Lemma 2.2.7. *For f_1, f_2, A formal series and $A \in \mathring{\mathcal{O}}(3)$*

$$\{\exp(\mathrm{ad}_A) f_1, \exp(\mathrm{ad}_A) f_2\} = \exp(\mathrm{ad}_A)\{f_1, f_2\}.$$

This can be proved by invoking a Borel summation and using Lemma 2.2.6.

2.2.2 Birkhoff normal form

We prove here a formal Birkhoff normal form for commuting Hamiltonians near a focus-focus singularity.

Theorem 2.2.8. *Let $f_1, f_2 \in \mathcal{E}$, such that*

– $\{f_1, f_2\} = 0$;

– $(f_1, f_2) = (q_1, q_2) \mod \mathcal{O}(3)$.

Then there exists $A \in \mathring{\mathcal{O}}(3)$, and there exists $g_i \in \mathbb{R}[\![t_1, t_2]\!]$, such that:

$$\exp(\mathrm{ad}_A)(f_i) = g_i(q_1, q_2), \qquad i = 1, 2. \tag{2.6}$$

Proof. First we remark that (2.6) is equivalent to saying that $\exp(\mathrm{ad}_A)(f_i)$ commutes (for the Poisson bracket) with q_1 and q_2. Indeed, let

$$\mathbf{z}_f^{\alpha\beta} = z_1^{\alpha_1} z_2^{\alpha_2} \bar{z}_1{}^{\beta_1} \bar{z}_2{}^{\beta_2}.$$

Using the formula for the flows of focus-focus components (2.3) we have

$$\{q_1, \mathbf{z}_f^{\alpha\beta}\} = (\alpha_1 - \alpha_2 + \beta_1 - \beta_2)\mathbf{z}_f^{\alpha\beta} \tag{2.7}$$

$$\{q_2, \mathbf{z}_f^{\alpha\beta}\} = i(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)\mathbf{z}_f^{\alpha\beta}. \tag{2.8}$$

Both brackets simultaneously vanish if and only if $\alpha_1 + i\alpha_2 = \beta_2 + i\beta_1$. In this case $\mathbf{z}^{\alpha\beta}$ is of the form $(\bar{z}_1 z_2)^\lambda (z_1 \bar{z}_2)^\mu = q^\lambda \bar{q}^\mu$, where $q = q_1 + iq_2 = \bar{z}_1 z_2$. Thus

$$\ker(\mathrm{ad}_{q_1}) \cap \ker(\mathrm{ad}_{q_2}) = \mathbb{R}[\![q_1, q_2]\!].$$

We may now turn to the proof of (2.6). We follow the usual proof by induction: if $A^{(N)} \in \mathcal{O}(3)$ is such that

$$\exp(\mathrm{ad}_{A^{(N)}})(f_i) \in \mathbb{R}[\![q_1, q_2]\!] + \mathcal{O}(N+1), \tag{2.9}$$

then we try to improve the error term to $\mathcal{O}(N+2)$ by replacing $A^{(N)}$ by $A^{(N+1)} := A^{(N)} + A_{N+1}$, with $A_{N+1} \in \mathcal{D}_{N+1}$. A standard calculation gives the so-called *cohomological equation* in $\mathcal{D}_{N+1}$, which here is a system:

$$\{q_i, A_{N+1}\} - R_{i,N+1} \in \mathcal{D}_{N+1} \cap \mathbb{R}[\![q_1, q_2]\!] \qquad i = 1, 2\,, \tag{2.10}$$

where $R_{i,N+1}$ is the error term of order $N+1$ in (2.9). Since $\{f_1, f_2\} = 0 = \{q_1, q_2\}$, we get from (2.9) the cocycle condition

$$\{q_1, R_{2,N+1}\} = \{q_2, R_{1,N+1}\}. \tag{2.11}$$

We see from (2.7) and (2.8) that the operator ad_{q_i} is diagonal in the basis $z^\alpha \bar{z}^\beta$, $\alpha, \beta \in \mathbb{N}^2$. Thus each space $\mathcal{D}_N$ is stable under ad_{q_i} and we have

$$\mathcal{D}_N = \ker \mathrm{ad}_{q_i} \oplus \mathrm{im}\, \mathrm{ad}_{q_i}, \quad i = 1, 2.$$

We denote by Π_j the linear projection onto $\ker \mathrm{ad}_{q_j}$, and by $\mathcal{L}_j = 1 - \Pi_j$ the projection onto the image. We let $S_j = \mathrm{ad}_{q_j}^{-1}\mathcal{L}_j$. Naturally, being diagonal in a common basis, the operators ad_{q_j}, Π_j, $\mathcal{L}_j$ and S_j, $j = 1, 2$, commute with each other. Now a simple calculation shows that the cohomological equation (2.10), which can be written as $\mathcal{L}_j(\mathrm{ad}_{q_j}(A_{N+1}) - R_j) = 0$, is solved by the following formula:

$$A_{N+1} = S_1 R_{1,N+1} + S_2 \Pi_1 R_{2,N+1}.$$

Of course, the cocycle condition $\mathrm{ad}_{q_1} R_2 = \mathrm{ad}_{q_2} R_1$ (cf (2.11)) is crucial in this computation. □

As a corollary to the Birkhoff normal form, we get a statement concerning smooth functions, up to a flat term.

Lemma 2.2.9. *Let $F = (f_1, f_2)$ whose Taylor series satisfy the same hypothesis as in the Birkhoff theorem 2.2.8. Then there exists a symplectomorphism χ of $\mathbb{R}^4$, tangent to the identity, and a smooth local diffeomorphism $\tilde{G} : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, tangent to the identity, such that:*

$$\chi^* F = \tilde{G}(q_1, q_2) + \mathcal{O}(\infty).$$

Proof. We use the notation of (2.6). Let $\tilde{g}_j$, $j = 1, \ldots, n$ be Borel summations of the formal series g_j, and let $\tilde{A}$ be a Borel summation of A. Let $\tilde{G} := (\tilde{g}_1, \ldots, \tilde{g}_n)$ and $\chi := \varphi^1_{\tilde{A}}$. Applying Lemma 2.2.6, we see that the Taylor series of $\chi^* F - \tilde{G} \circ (q_1, q_2)$ is flat at the origin.

□

Lemma 2.2.9 gives us the main theorem modulo a flat function. The rest of the paper is devoted to absorbing this flat function.

2.3 An equivariant flat Morse lemma

One of the key ingredients of the proof is a (smooth, but non symplectic) equivariant Morse lemma for commuting functions. In view of the Birkhoff normal form, it is enough to state it for flat perturbations of quadratic forms, as follows.

2.3.1 A flat Morse lemma

We start by a simple (non equivariant) version of the Morse lemma for focus-focus singularities with flat remainder.

Theorem 2.3.1. *Let h_1, h_2 be $\mathcal{C}^\infty$ functions defined near the origin in $\mathbb{R}^4$, such that*

$$\text{For } i = 1, 2, \qquad h_i = q_i + \mathcal{O}(\infty).$$

Then there exists a local diffeomorphism near the origin in $\mathbb{R}^4$, of the form $\Upsilon = id + \mathcal{O}(\infty)$, such that

$$\Upsilon^* h_i = q_i \quad i = 1, 2 \qquad \text{on } \Omega.$$

Proof. Using Moser's path method, we shall look for Υ as the time-1 flow of a time-dependent vector field X_t, which should be uniformly flat for $t \in [0, 1]$.

To simplify notations, let $Q := (q_1, q_2)$. Let $H := (h_1, h_2)$ and consider the interpolation $H_t := (1-t)Q + tH$. We want X_t to satisfy

$$(\varphi^t_{X_t})^* H_t = Q \ , \ \forall t \in [0, 1].$$

Differentiating this equation with respect to t, we get

$$(\varphi^t_{X_t})^* \left[\frac{\partial H_t}{\partial t} + \mathcal{L}_{X_t} H_t \right] = (\varphi^t_{X_t})^* \left[-Q + H + (\imath_{X_t} d + d\imath_{X_t}) H_t \right] = 0.$$

So it is enough to find a neighbourhood of the origin where one can solve, for $t \in [0, 1]$, the system of equations

$$dH_t(X_t) = Q - H =: R. \tag{2.12}$$

By assumption R is flat and $dH_t = dQ + tdR = dQ + \mathcal{O}(\infty)$. Let Ω be an open neighborhood of the origin in $\mathbb{R}^4$. Let us consider dQ as a linear operator from $\chi(\Omega)$, the space of smooth vector fields, to $\mathcal{C}^\infty(\Omega) \times \mathcal{C}^\infty(\Omega)$. This operator sends flat vector fields to flat functions.

The discussion is easier if one works with the appropriate topology on flat functions. Assume Ω is contained in the euclidean unit ball and contains 0. Let $\mathcal{F}\ell^\infty(\Omega)$ denote the vector space of smooth functions defined on Ω and flat in 0. For each integer $N \geqslant 0$, and each $f \in \mathcal{F}\ell^\infty(\Omega)$, the quantity

$$p_N(f) = \sup_{z \in \Omega} \frac{|f(z)|}{\|z\|_2^N}$$

is finite due to (2.4), and thus the family (p_N) is an increasing[1] family of norms on $\mathcal{F}\ell^\infty(\Omega)$. We call the corresponding topology the "local topology at the origin", as opposed to the usual topology defined by suprema on compact subsets of Ω. Thus, a linear operator A from $\mathcal{F}\ell^\infty(\Omega)$ to itself is continuous in the local topology if and only if

$$\forall N \geq 0, \quad \exists N' \geq 0, \exists C > 0, \forall f \quad p_N(Af) \leq C p_{N'}(f). \tag{2.13}$$

For such an operator, if f depends on an additional parameter and is uniformly flat, in the sense that the estimates (2.4) are uniform with respect to that parameter, then Af is again uniformly flat.

1. $p_{N+1} \geq p_N$

Lemma 2.3.2. *Restricted to flat vector fields and flat functions, dQ admits a linear right inverse $\Psi : \mathcal{F}\ell^\infty(\Omega)^2 \to \chi(\Omega)_{flat}$. For every $U = (u_1, u_2) \in \mathcal{C}^\infty(\Omega)^n$ with $u_j \in \mathcal{O}(\infty)$, one has*

$$dq_i(\Psi(U)) = u_i.$$

Moreover Ψ is continuous in the local topology.

Now assume that the lemma holds, and let $A := \Psi \circ dR$, where $R = (r_1, r_2)$ was defined in (2.12). The operator A is linear and goes from $\chi(\Omega)_{\text{flat}}$ to itself, sending a vector field v to the vector field $\Psi(dr_1(v), dr_2(v))$. From the lemma, we get:

$$dQ(A(v)) = dR(v).$$

We claim that for Ω small enough, the operator $(\text{Id} - tA)$ is invertible, and its inverse is continuous in the local topology, uniformly for $t \in [0, 1]$. Now let

$$X_t := (\text{Id} - tA)^{-1} \circ \Psi(R).$$

We compute:

$$dH_t(X_t) = dQ(X_t) - tdR(X_t) = dQ(X_t) - tdQ(A(X_t)) = dQ(\text{Id} - tA)(X_t).$$

Hence

$$dH_t(X_t) = dQ(\Psi(R)) = R.$$

Thus equation (2.12) is solved on Ω. Since $X_t(0) = 0$ for all t, the standard Moser's path argument shows that, up to another shrinking of Ω, the flow of X_t is defined up to time 1. Because of the continuity in the local topology, X_t is uniformly flat, which implies that the flow at time 1, Υ, is the identity modulo a flat term.

□

To make the above proof complete, we still need to prove Lemma 2.3.2 and the claim concerning the invertibility of $(\text{Id} - tA)$.

Proof of Lemma 2.3.2. Consider the focus-focus model $q := q_1 + iq_2$ on $\mathbb{R}^4$. Given a flat function $u := u_1 + iu_2$, we want to find a real vector field Y such that

$$dq(Y) = u. \tag{2.14}$$

We look for Y in the form $Y = a\frac{\partial}{\partial z_1} + b\frac{\partial}{\partial \bar{z}_1} + c\frac{\partial}{\partial z_2} + d\frac{\partial}{\partial \bar{z}_2}$. The vector field Y is real if and only if $a = \bar{b}$ and $c = \bar{d}$. Writing

$$dq = d(\bar{z}_1 z_2) = z_2 d\bar{z}_1 + \bar{z}_1 dz_2$$

we see that (2.14) is equivalent to $z_2 b + \bar{z}_1 c = u$. Since u is flat, there exists a smooth flat function $\tilde{u}$ such that $u = (|z_1|^2 + |z_2|^2)\tilde{u} = z_1\bar{z}_1\tilde{u} + z_2\bar{z}_2\tilde{u}$.

Thus simply take $b = \bar{z}_2\tilde{u}$ and $c = z_1\tilde{u}$. Back in the basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2})$, we get

$$Y = \frac{1}{|z_1|^2 + |z_2|^2} \left(\Re e(\bar{z}_2 u), \Im m(\bar{z}_2 u), \Re e(z_1 u), \Im m(z_1 u)\right)$$

Thus, Ψ is indeed a linear operator and its matrix in this basis is

$$\frac{1}{x_1^2 + x_2^2 + \xi_1^2 + \xi_2^2} \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \\ x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix} \tag{2.15}$$

Finally, if $|u_j| \leq C(x_1^2 + x_2^2 + \xi_1^2 + \xi_2^2)^{N/2}$ for $j = 1, 2$, then

$$\|\Psi(U)(x_1, x_2, \xi_1, \xi_2)\| \leq d(\Omega) C (x_1^2 + x_2^2 + \xi_1^2 + \xi_2^2)^{N/2-1}. \tag{2.16}$$

Here $\|\Psi(U)(x_1, x_2, \xi_1, \xi_2)\|$ is the supremum norm in $\mathbb{R}^4$ and $d(\Omega)$ is the diameter of Ω. Thus Ψ is continuous in the local topology. $\square$

Now consider the operator $A = \Psi \circ dR$. We see from equation (2.15) and the flatness of the partial derivatives of R that, when expressed in the basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2})$ of $\chi(\Omega)_{\text{flat}}$, A is a 4×4 matrix with coefficients in $\mathcal{F}\ell^\infty(\Omega)$. In particular one may choose Ω small enough such that $\sup_{z\in\Omega} \|A(z)\| < 1/2$. Thus, for all $t \in [0, 1]$, the matrix $(\mathrm{Id} - tA)$ is invertible and its inverse is of the form $\mathrm{Id} + t\tilde{A}_t$, where $\tilde{A}_t$ has smooth coefficients and $\sup_{z\in\Omega} \left\|\tilde{A}_t(z)\right\| < 1$.

Now $p_N(f + t\tilde{A}_t f) \leq p_N(f) + p_N(\tilde{A}_t f) \leq 2p_N(f)$: the linear operator $(\mathrm{Id} + t\tilde{A}_t)$ is uniformly continuous in the local topology. With this the proof of Theorem 2.3.1 is complete.

2.3.2 The equivariant flat Morse lemma

The Hamiltonian q_2 induces an S^1 action on $\mathbb{R}^4$. We prove here that the diffeomorphism Υ given by Theorem 2.3.1 can be made equivariant, provided that $\{h_1, h_2\} = 0$.

Theorem 2.3.3. *Let $H = (h_1, h_2)$ such that $\{h_1, h_2\} = 0$ and $H = Q + \mathcal{O}(\infty)$, with $Q = (q_1, q_2)$. Then there exists a local diffeomorphism near the origin in $\mathbb{R}^4$, of the form $\Upsilon = id + \mathcal{O}(\infty)$, and a local diffeomorphism V defined near $H(0) \in \mathbb{R}^2$ such that*

$$\Upsilon^*(V \circ H) = Q,$$

and Υ preserves the S^1-action generated by q_2:

$$\Upsilon^* \mathcal{X}^0_{q_2} = \mathcal{X}^0_{q_2}, \tag{2.17}$$

where $\mathcal{X}^0_{q_2}$ is the symplectic gradient of q_2 for the standard symplectic form ω_0.

The main ingredient will be the construction of a smooth Hamiltonian S^1-action on the symplectic manifold $(\mathbb{R}^4, \omega_0)$ that leaves the original moment map (h_1, h_2) invariant (this idea was extensively used by Zung in his papers. See [Zun06b] for a review of his work). The natural idea is to define the moment map H through an action integral on the Lagrangian leaves, but because of the singularity, it is not obvious that we get a smooth function.

Let γ_z be the loop in $\mathbb{R}^4$ equal to the S^1-orbit of z for the action generated by q_2 with canonical symplectic form ω_0. Explicitly (see (2.3)), we can write $z = (z_1, z_2)$ and

$$\gamma_z(t) = (e^{2\pi it} z_1, e^{2\pi it} z_2), \qquad t \in [0, 1].$$

Notice that if $z = 0$ then the "loop" γ_z is in fact just a point. A key formula is

$$q_2 = \frac{1}{2\pi} \int_{\gamma_z} \alpha_0.$$

This can be verified by direct computation, or as a consequence of the following lemma.

Lemma 2.3.4. *Let α_0 be the Liouville 1-form on $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ (thus $\omega_0 = d\alpha_0$). If I is a Hamiltonian which is homogeneous of degree n in the variables $(\xi_1, \ldots, \xi_n)$, defining $\mathcal{X}_I^0$ as the symplectic gradient of I induced by ω_0, we have :*

$$\alpha_0(\mathcal{X}_I^0) = nI.$$

Proof. Consider the $\mathbb{R}_+^*$-action on $T^*\mathbb{R}^n$ given by the multiplication on the cotangent fibers: $\varphi^t(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) = (x_1, \ldots, x_n, t\xi_1, \ldots, t\xi_n)$. Since

$$\alpha_0 = \sum_{i=1}^n \xi_i dx_i,$$

we have : $\varphi^{t*}\alpha_0 = t\alpha_0$. Taking the derivative with respect to t gives $\mathcal{L}_\Xi \alpha_0 = \alpha_0$ where Ξ is the infinitesimal action of φ^t : $\Xi = (0, \ldots, 0, \xi_1, \ldots, \xi_n)$. By Cartan's formula, $\imath_\Xi d\alpha_0 + d(\imath_\Xi \alpha_0) = \alpha_0$. But

$$\imath_\Xi \alpha_0 = \sum_{i=1}^n \xi_i dx_i(\Xi) = 0 \text{ , so } \alpha_0 = \imath_\Xi \omega_0.$$

Thus, $\alpha_0(\mathcal{X}_H^0) = \omega_0(\Xi, \mathcal{X}_H^0) = dH(\Xi)$, and since H is a homogeneous function of degree n with respect to Ξ, Euler's formula gives

$$\mathcal{L}_\Xi H = dH(\Xi) = nH.$$

Therefore we get, as required:

$$dH(\Xi) = nH(\Xi).$$

□

From this lemma, we deduce, since q_2 is invariant under $\mathcal{X}^0_{q_2}$:

$$\frac{1}{2\pi}\int_{\gamma_z}\alpha_0 = \int_0^1 \alpha_{0\gamma_2(t)}(\mathcal{X}^0_{q_2})dt = \int_0^1 q_2(\gamma_z(t)) = q_2.$$

By the *classic* flat Morse lemma we have a local diffeomorphism $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\Phi^* h_j = q_j$, $j = 1, 2$. Let $\alpha := (\Phi^{-1})^*\alpha_0$, and let

$$K(z) := \frac{1}{2\pi}\int_{\gamma_z}\alpha,$$

and let $I := K \circ \Phi$. Note that $I(m) = \int_{\tilde{\gamma}_m}\alpha_0$ where $\tilde{\gamma}_m := \Phi^{-1} \circ \gamma_{z=\Phi(m)}$, and $I(0) = 0$.

Lemma 2.3.5. *The function I is in $\mathcal{C}^\infty(\mathbb{R}^4, 0)$.*

Proof. Equivalently, we prove that $K \in \mathcal{C}^\infty(\mathbb{R}^4, 0)$. The difficulty lies in the fact that the family of "loops" γ_z degenerates into a point when $z = 0$. However it is easy to desingularize K as follows. We identify $\mathbb{R}^4$ with $\mathbb{C}^2$, and we introduce the maps:

$$\begin{array}{rclcrcl} j : \mathbb{D} \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 & \quad & j_z : \mathbb{D} & \to & M \\ (\zeta, (z_1, z_2)) & \mapsto & (\zeta z_1, \zeta z_2) & & \zeta & \mapsto & (\zeta z_1, \zeta z_2) \end{array}$$

so that $\gamma_z = (j_z)_{\restriction U(1)}$. Let $D \subset \mathbb{C}$ be the closed unit disk $\{\zeta \le 1\}$. Thus

$$\int_{\gamma_z}\alpha = \int_{j_z(U(1))}\alpha = \int_{U(1)} j_z^*\alpha = \int_{\partial D} j_z^*\alpha.$$

Let $\omega := d\alpha$. By Stokes' formula,

$$\int_{\partial D} j_z^*\alpha = \int_D j_z^*\omega = \int_D \omega_{j(z,\zeta)}(d_\zeta j(z,\zeta)(\cdot), d_\zeta j(z,\zeta)(\cdot)).$$

Since D is a fixed compact set and ω, j are smooth, we get $K \in \mathcal{C}^\infty(\mathbb{R}^4, 0)$. $\square$

Consider now the integrable system (h_1, h_2). Since I is an action integral for the Liouville 1-form α_0, associated to the torus foliation given by (h_1, h_2), it follows from the action-angle theorem of Liouville-Arnold-Mineur that the Hamiltonian flow of I preserves the regular Liouville tori of (h_1, h_2). Thus, $\{I, h_j\} = 0$ for $j = 1, 2$ on every regular torus. The function $\{I, h_j\}$ being smooth hence continuous, $\{I, h_j\} = 0$ everywhere it is defined: I is locally constant on every level set of the joint moment map (h_1, h_2). Equivalently, K is locally constant on the level sets of $q = (q_1, q_2)$. It is easy to check that these level sets are locally connected near the origin. Thus there exists a map $g : (\mathbb{R}^2, 0) \to \mathbb{R}$ such that

$$K = g \circ q.$$

It is easy to see that g must be smooth: indeed, K itself is smooth and, in view of (2.1), one can write

$$g(c_1, c_2) = K(x_1 = c_1, x_2 = -c_2, \xi_1 = 1, \xi_2 = 0). \tag{2.18}$$

We claim that the function $(c_1, c_2) \mapsto g(c_1, c_2) - c_2$ is flat at the origin: since $\Phi = id + \mathcal{O}(\infty)$, we have: $\alpha = \Phi^* \alpha_0 = \alpha_0 + \mathcal{O}(\infty)$ so $K(z) = \int_{\gamma_2[z]} \alpha_0 + \mathcal{O}(\infty) = q_2 + \eta(z)$ with η a flat function of the 4 variables. We show now the lemma:

Lemma 2.3.6. *Let $\eta \in \mathcal{C}^\infty(\mathbb{R}^4; \mathbb{R})$ be a flat function at the origin in $\mathbb{R}^4$ such that $\eta(z) = \mu(q_1, q_2)$ for some map $\mu : \mathbb{R}^2 \to \mathbb{R}$. Then μ is flat at the origin in $\mathbb{R}^2$.*

Proof. We already know from (2.18) that μ has to be smooth. Since η is flat, we have, for some constant C_N,

$$|\eta(x_1, x_2, \xi_1, \xi_2)| \leq C_N \left\|(x_1, x_2, \xi_1, \xi_2)\right\|^N .$$

But for any $c = (c_1 + ic_2) \in \mathbb{C}$, there exists $(z_1, z_2) \in q^{-1}(c)$ such that $2\left|c\right|^2 = \left|z_1\right|^2 + \left|z_2\right|^2$: if $c = 0$ we take $z = 0$, otherwise take $z_1 := |c|^{1/2}$ and $z_2 := c/z_1$, so that $q(z_1, z_2) = \bar{z}_1 z_2 = c$. Therefore, for all $(c_1, c_2) \in \mathbb{R}^2$ we can write

$$|\mu(c_1, c_2)| = |\eta(z_1, z_2)| \leq C_N \|(z_1, z_2)\|^N \leq 2C_N |c|^N,$$

which finishes the proof. ☐

We have now :

$$\begin{pmatrix} h_1 \\ I \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 + \mu(h_1, h_2) \end{pmatrix}$$

By the implicit function theorem, the function $V : (x, y) \mapsto (x, y + \mu(x, y))$ is locally invertible around the origin; moreover, V^{-1} is infinitely tangent to the identity. Therefore, in view of the statement of Theorem 2.3.3, we can replace our initial integrable system (h_1, h_2) by the system $V \circ (h_1, h_2) = (h_1, I)$.

Thus, we have reduced our problem to the case where $h_2 = I$ is a Hamiltonian for a smooth S^1 action on $\mathbb{R}^4$. We denote by S^1_I this action. The origin is a fixed point, and we denote by $\mathrm{lin}(S^1_I)$ the action linearized at the origin. We now invoke an equivariant form of the Darboux theorem.

Theorem 2.3.7 (Chaperon-Darboux-Weinstein, [Cha86])**.** *There exists φ a diffeomorphism of $(\mathbb{R}^4, 0)$ such that :*

$$\left(\mathbb{R}^4, \omega_0, S^1_I\right) \xrightarrow{\varphi} \left(T_0\mathbb{R}^4, T_0\omega_0, \mathrm{lin}(S^1_I)\right)$$

The linearization $T_0\omega_0$ of ω_0 is ω_0 : φ is a symplectomorphism, and the linearization of the S^1-action of I is the S^1-action of the quadratic part of I, which is q_2. Hence $I \circ \varphi^{-1}$ and q_2 have the same symplectic gradient, and both vanish at the origin: so $I \circ \varphi^{-1} = q_2$. So we have got rid of the flat part of h_2 without modifying the symplectic form. The last step is to apply a *precised* version of the equivariant flat Morse lemma in our particular case:

Lemma 2.3.8. *Let h_1, h_2 be functions in $\mathcal{C}^\infty(\mathbb{R}^4, 0)$ such that $\{h_1, h_2\} = 0$ and*

$$\begin{cases} h_1 = q_1 + \mathcal{O}(\infty) \\ h_2 = q_2. \end{cases}$$

Then there exists a local diffeomorphism Υ of $(\mathbb{R}^4, 0)$ of the form $\Upsilon = id + \mathcal{O}(\infty)$ such that

$$\Upsilon^* h_i = q_i \ , \ i = 1, 2$$

and

$$\Upsilon^* \mathcal{X}^0_{q_2} = \mathcal{X}^0_{q_2}.$$

Proof. Following the same Moser's path method we used in the proof of the classical flat Morse lemma, we come up with the following cohomological equation (2.12)

$$(Z) \begin{cases} (dq_1 + tdr_1)(X_t) = r_1 \\ dq_2(X_t) = 0 \end{cases}$$

Theorem 2.3.1 ensures the existence of a solution X_t to this system. We have then that $\{r_1, q_2\} = \{r_2, q_1\} = 0$, because here $r_2 = 0$. We have also that : $\{q_1, q_2\} = 0$, $\{q_2, q_2\} = 0$, so r_1,q_1 and q_2 are invariant by the flow of q_2. So we can average (Z) by the action of q_2: let $\varphi_2^s := \varphi^s_{\mathcal{X}^0_{q_2}}$ be the time-s flow of the vector field $\mathcal{X}^0_{q_2}$ and let

$$\langle X_t \rangle := \frac{1}{2\pi} \int_0^{2\pi} (\varphi_2^s)^* X_t ds.$$

If a function f is invariant under φ_2^s, *i.e.* $(\varphi_2^s)^* f = f$, then

$$((\varphi_2^s)^* X_t) f = ((\varphi_2^s)^* X_t)((\varphi_2^s)^* f) = (\varphi_2^s)^*(X_t f).$$

Integrating over $s \in [0, 2\pi]$, we get $\langle X_t \rangle f = \langle X_t f \rangle$, where the latter is the standard average of functions. Therefore $\langle X_t \rangle$ satisfies the system (Z) as well.

Finally, we have, for any s, $(\varphi_2^t)^* \langle X_t \rangle = \langle X_t \rangle$ which implies

$$[\mathcal{X}^0_{q_2}, \langle X_t \rangle] = 0; \tag{2.19}$$

in turn, if we let $\varphi^t_{\langle X_t \rangle}$ be the flow of the non-autonomous vector field $\langle X_t \rangle$, integrating (2.19) with respect to t gives $(\varphi^t_{\langle X_t \rangle})^* \mathcal{X}^0_{q_2} = \mathcal{X}^0_{q_2}$. For $t = 1$ we get $\Upsilon^* \mathcal{X}^0_{q_2} = \mathcal{X}^0_{q_2}$. Notice that, by naturality $\Upsilon^* \mathcal{X}^0_{q_2}$ is the symplectic gradient of $\Upsilon^* q_2 = q_2$ with respect to the symplectic form $\Upsilon^* \omega_0 = \omega$. $\square$

2.4 Principal lemma

2.4.1 Division lemma

The following cohomological equation, formally similar to (2.10), is the core of Theorem 2.1.2. It was proven in [MVuN05, Proposition 4.3,point (3)].

Theorem 2.4.1. *Let $r_1, r_2 \in \mathcal{C}^\infty((\mathbb{R}^4, 0); \mathbb{R})$, flat at the origin such that $\{r_1, q_2\} = \{r_2, q_1\}$.*

Then there exists $f \in \mathcal{C}^\infty((\mathbb{R}^4, 0); \mathbb{R})$ and $\phi_2 \in \mathcal{C}^\infty((\mathbb{R}^2, 0); \mathbb{R})$ such that

$$\begin{cases} \{f, q_1\}(x, \xi) = r_1 \\ \{f, q_2\}(x, \xi) = r_2 - \phi_2(q_1, q_2), \end{cases} \tag{2.20}$$

and f and ϕ_2 are flat at the origin. Moreover ϕ_2 is unique and given by

$$\forall z \in \mathbb{R}^4, \quad \phi_2(q_1(z), q_2(z)) = \frac{1}{2\pi} \int_0^{2\pi} (\varphi_{q_2}^s)^* r_2(z) ds, \tag{2.21}$$

where $s \mapsto \varphi_{q_2}^s$ is the Hamiltonian flow of q_2.

One can compare the difficulty to solve this equation in the 1D hyperbolic case with elliptic cases. If the flow is periodic, that is, if $SO(q)$ is compact, then one can solve the cohomological equation by averaging over the action of $SO(q)$. This is what happens in the elliptic case. But for a hyperbolic singularity, $SO(q)$ is not compact anymore, and the solution is more technical (see [dVV79]). In our focus-focus case, we have to solve simultaneously two cohomological equations, one of which yields a compact group action while the other doesn't. This time again, it is the "cross-commuting relation" $\{r_i, q_j\} = \{r_j, q_i\}$ that we already encountered in the formal context that will allow us to solve simultaneously both equations.

2.4.2 A Darboux lemma for focus-focus foliations

Here again $\mathbb{R}^4$ is endowed with the canonical symplectic form ω_0. Recall that the regular level sets of the map $q = (q_1, q_2) : \mathbb{R}^4 \to \mathbb{R}^2$ are Lagrangian for ω_0.

Proposition 2.4.2. *Let ω be a symplectic form on $\mathbb{R}^4$ such that*

(a) $\omega = \omega_0 + \mathcal{O}(\infty)$;

(b) the regular level sets of q are Lagrangian for ω;

(c) for all $z \in \mathbb{R}^4$,

$$\int_{D_z} \omega - \omega_0 = 0,$$

where D_z is the disk given by

$$D_z := \{(\zeta z_1, \zeta z_2) \in \mathbb{C}^2; \quad \zeta \in \mathbb{C}, |\zeta| \le 1\}$$

(here we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and denote $z = (z_1, z_2) \in \mathbb{C}^2$).

Then there exists a local diffeomorphism Φ *of* $(\mathbb{R}^4, 0)$ *and* U *a local diffeomorphism of* $(\mathbb{R}^2, 0)$ *such that*

1. $\Phi^*\omega = \omega_0$
2. $q \circ \Phi = U \circ q$
3. *Both* Φ *and* U *are infinitely tangent to the identity.*

Notice that condition 2. means that Φ preserves the (singular) foliation defined by the level sets of q. Notice also that the hypothesis (a),(b),(c) are in fact necessary: for (a) and (b) this is obvious; for (c), remark that $\gamma_z := \partial D_z$ is an orbit of the S^1-action generated by q_2 for the canonical symplectic form ω_0, and thus is a homology cycle on the Liouville torus $q = \text{const}$. Since Φ is tangent to the identity, $\Phi_*\gamma_z$ is homologous to γ_z when z is small enough; thus, if α_0 is the Liouville 1-form on $\mathbb{R}^4$, which is closed on the Liouville tori, we have

$$\int_{\gamma_z} \alpha_0 = \int_{\Phi_*(\gamma_z)} \alpha_0 = \int_{\gamma_z} \Phi^*\alpha_0,$$

which by Stokes gives (c).

Proof of the proposition. We use again the standard deformation method by Moser. Let

$$\omega_s = (1-s)\omega_0 + s\omega.$$

We look for Y_s a time-dependant vector field defined for $s \in [0,1]$ whose flow $s \mapsto \varphi^s_{Y_s}$ satisfies $(\varphi^s_{Y_s})^*\omega_s = \omega_0$. Taking the derivative with respect to s gives

$$(\varphi^s_{Y_s})^* \left[\frac{\partial \omega_s}{\partial s} + \mathcal{L}_{Y_s}\omega_s \right] = (\varphi^s_{Y_s})^* \left[\omega - \omega_0 + d(\imath_{Y_s}\omega_s) \right] = 0.$$

ω and ω_0 being closed, we can find, in a neighbourhood of the origin, smooth 1-forms α and α_0 such that $\omega = d\alpha$ and $\omega_0 = d\alpha_0$. Using the standard constructive proof of the Poincaré lemma, we can choose α and α_0 such that $\alpha = \alpha_0 + \mathcal{O}(\infty)$. Let $\varphi^t_{q_2}$ be the Hamiltonian flow of X^0_2 on $(\mathbb{R}^4, \omega_0)$.

Since $\omega_s(0) = \omega_0(0) = \omega_0$, one can find a neighbourhood of the origin on which ω_s is non-degenerate for all $s \in [0,1]$. This enables us to find a suitable Y_s by solving

$$\imath_{Y_s}\omega_s = -(\alpha - \alpha_0) + df, \tag{2.22}$$

for a suitable function f. Here, any function f such that $df(0) = 0$ will yield a vector field Y_s whose time-1 flow Φ solves the point 1. of the lemma. It turns out that properly choosing f will be essential in ensuring that Φ preserves the foliation (point 2. and 3.).

Let X^0_1, X^0_2 be the Hamiltonian vector fields associated to q_1, q_2 respectively, for ω_0. Since the level sets of q are Lagrangian for ω_0, X^0_1, X^0_2 are commuting vector fields spanning the tangent space to regular leaves. Thus $\omega_0(X^0_1, X^0_2) = 0$. But, by assumption, the level sets of q are Lagrangian for ω

as well. This implies that $\omega(X_1^0, X_2^0) = 0$ as well. Thus $\omega_s(X_1^0, X_2^0) = 0$ for all s: the level sets of q are Lagrangian for ω_s. This entails that the condition that Y_s be tangent to the leaves can be written

$$\begin{cases} \omega_s(Y_s, X_1^0) = 0 \\ \omega_s(Y_s, X_2^0) = 0. \end{cases} \tag{2.23}$$

We can expand this:

$$(2.23) \Longleftrightarrow_{(2.22)} \begin{cases} -(\alpha - \alpha_0)(X_1^0) + df(X_1^0) = 0 \\ -(\alpha - \alpha_0)(X_2^0) + df(X_2^0) = 0. \end{cases}$$

Now we may let

$$\begin{cases} r_1 := (\alpha - \alpha_0)(X_1^0) \\ r_2 := (\alpha - \alpha_0)(X_2^0) \end{cases}$$

and the condition becomes

$$(2.23) \Longleftrightarrow \begin{cases} \{f, q_1\} = r_1 \\ \{f, q_2\} = r_2. \end{cases}$$

(Here the Poisson brackets come from the canonical symplectic form ω_0). Notice that r_1 and r_2 are flat at the origin. Next, recall the following formula for 1-forms :

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]).$$

Thus

$$0 = \omega_0(X_1^0, X_2^0) = d\alpha_0(X_1^0, X_2^0) = X_1^0\alpha_0(X_2^0) - X_2^0\alpha_0(X_1^0) - \alpha_0([X_1^0, X_2^0]),$$

which implies

$$\imath_{X_1^0} d(\alpha_0(X_2^0)) = \imath_{X_2^0} d(\alpha_0(X_1^0)),$$

and similarly we have

$$\imath_{X_1^0} d(\alpha(X_2^0)) = \imath_{X_2^0} d(\alpha(X_1^0)).$$

Hence we may write the same equation again for $\alpha - \alpha_0$ which, in terms of ω_0-Poisson brackets, becomes

$$\{r_1, q_2\} = \{r_2, q_1\}.$$

Therefore, a solution f to this system (2.23) is precisely given by the division lemma (Theorem 2.4.1), provided we show that r_2 has vanishing $\varphi_{q_2}^t$-average. But, since $\frac{d}{dt}\varphi_{q_2}^t = X_2^0(\varphi_{q_2}^t)$, we have,

$$\forall z \in \mathbb{R}^4, \quad \frac{1}{2\pi}\int_0^{2\pi} r_2(\varphi_{q_2}^t(z))dt = \int_{\gamma_z} \alpha - \alpha_0 = \int_{D_z} \omega - \omega_0 = 0.$$

Finally, we check that Y_s as defined with (2.22) vanishes at the origin and hence yields a flow up to time 1 on a open neighbourhood of the origin.

To conclude, the time-1 flow of Y_s is a local diffeomorphism Φ that preserves the q-foliation and such that $\Phi^*\omega = \omega_0$, which finishes the proof.

Notice that Y_s is uniformly flat, whence Φ is a flat perturbation of the identity. $\square$

2.5 Proof of the main theorem

Theorem 2.1.2 follows from successively applying Lemma 2.2.9, Theorem 2.3.3, and Proposition 2.4.2. We summarize the different steps of the proof in the diagram below

$$\begin{cases} F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ \omega_0 \end{cases} \xrightarrow{\text{Lemma 2.2.9}} \begin{cases} \chi^* F = G(q_1, q_2) + \mathcal{O}(\infty) \\ \chi^*\omega_0 = \omega_0, \end{cases}$$

$$\xrightarrow{\text{Theorem 2.3.3}} \begin{cases} \Upsilon^* G^{-1}(\chi^* F) = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ \Upsilon^*(\chi^*\omega_0) = \omega = \omega_0 + \mathcal{O}(\infty), \end{cases}$$

$$\xrightarrow{\text{Proposition 2.4.2}} \begin{cases} \Phi^*\Upsilon^* G^{-1}(\chi^* F) = \begin{pmatrix} U(q_1, q_2) \\ U(q_1, q_2) \end{pmatrix} \\ \Phi^*\Upsilon^*\chi^*\omega_0 = \omega_0. \end{cases}$$

Only the last implication needs an explanation: indeed, even if the Morse lemma (Theorem 2.3.3) is not symplectic, the initial foliation by F is Lagrangian for ω_0, and this implies that, under Υ, the target foliation by q becomes Lagrangian for the target symplectic form ω. Thus the hypotheses (a) and (b) of the Darboux lemma (Proposition 2.4.2) are satisfied. That (c) is also satisfied follows from the equivariance property of Theorem 2.3.3. Indeed, let α_0 be the Liouville 1-form in $\mathbb{R}^4$, and $\alpha := \Upsilon^*\alpha_0$. Since Υ commutes with $\varphi^t_{q_2}$, we have

$$\mathcal{L}_{X_2^0}\alpha = \mathcal{L}_{X_2^0}\Upsilon^*\alpha_0 = \Upsilon^*\mathcal{L}_{X_2^0}\alpha_0.$$

On the other hand, since $\imath_{X_2^0} d\alpha_0 = -dq_2$ and (Lemma 2.3.4) $d\imath_{X_2^0}\alpha_0 = dq_2$, we get $\mathcal{L}_{X_2^0}\alpha_0 = 0$. Thus $\mathcal{L}_{X_2^0}\alpha = 0$ which, in turn, says that $d\imath_{\mathcal{X}^\omega_{q_2}}\alpha = -\imath_{\mathcal{X}^\omega_{q_2}} d\alpha = dq_2$, where we denote by $\mathcal{X}^\omega_{q_2}$ the ω-gradient of q_2. By property (2.17), $\mathcal{X}^\omega_{q_2} = X_2^0$, so $d\imath_{X_2^0}\alpha = dq_2$. Hence $\imath_{X_2^0}\alpha = q_2 + \beta$, where β is a constant, which is actually equal to 0 since $\imath_{X_2^0}\alpha = q_2 + \mathcal{O}(\infty)$. We thus get $\imath_{X_2^0}(\alpha - \alpha_0) = 0$, which of course implies

$$\int_{\gamma_z} \alpha - \alpha_0 = 0.$$

Thus one may apply Proposition 2.4.2, and the main theorem 2.1.2 is shown for $\Psi := \Phi \circ \Upsilon \circ \chi$ and $\tilde{G} := G \circ U$.

Chapter 3

Local models of semi-toric integrable systems

3.1 Local models of orbits and leaves

3.1.1 Semi-local normal form

For a Hamiltonian system, the orbit $\mathcal{O}_m$ of a critical point $m \in M$ by the local Poisson $\mathbb{R}^n$-action is a submanifold of dimension equal to the rank k_{x}. For this subsection, we can assume without loss of generality that $df_1 \wedge \cdots \wedge df_{k_{\mathrm{x}}} \neq 0$.

Definition 3.1.1. *The orbit $\mathcal{O}_m$ is called non-degenerate if, when we take the symplectic quotient of a neighborhood of $\mathcal{O}_m$ by the Poisson action of $\mathbb{R}^{k_{\mathrm{x}}}$ generated by $F_{\mathrm{x}} = (f_1, \ldots, f_{k_{\mathrm{x}}})$, the image of m is a non-degenerate fixed point. A leaf is called non-degenerate if it has only non-degenerate points.*

A non-degenerate orbit has only non-degenerate critical points of the same Williamson index. Thus it makes sense to talk of an orbit, or even of a regular leaf of a given Williamson index. The linear model of a non-degenerate orbit is the same as the linear model of a point. Of course, a non-degenerate Hamiltonian system has only non-degenerate orbits and non-degenerate leaves. Non-degeneracy is an open property.

Theorem 3.1.2 (Miranda & Zung, [MZ04])**.** *Let m be a non-degenerate critical point of Williamson type $\mathbb{k}$. There exists a neighborhood $\tilde{\mathcal{U}}_m$ saturated with respect to the action of F_{x}, the transverse components of F, a symplectic group action of a finite group H_0 and a symplectomorphism:*

$$\varphi : (\tilde{\mathcal{U}}_m, \omega) \to \varphi(\tilde{\mathcal{U}}_m) \subset L_{\mathbb{k}}/H_0$$

such that:

– $\varphi^* F \sim Q_{\mathbb{k}}$,

– *The transverse orbit $\mathcal{O}_{F_\mathrm{x}}(m)$ is sent to the zero-torus*

$$\mathcal{T} = \{\mathbf{x}^{\mathrm{e,f,h}} = \boldsymbol{\xi}^{\mathrm{e,f,h}} = 0\ ,\ \mathbf{I} = 0\}$$

of dimension k_x.

Moreover, if there exists a symplectic action of a compact group $G \circlearrowright M$ that preserves the moment map F, the action of F can be linearized equivariantly with respect to that group action.

Remark 3.1.3. *The finite group action H_0 is trivial in particular if $k_\mathrm{h} = 0$ and $k_\mathrm{f} = 1$, that is if we have a semi-toric integrable system.*

This theorem is an extension of Eliasson's normal form theorem 1.2.23: it means we can linearize the singular Lagrangian foliation of an integrable system symplectically on an orbital neighborhood of a non-degenerate critical point.

We shall call this result an "orbital", or "semi-local" result, as this normal form is valid in a neighborhood that is larger than an ε-ball of a critical point, as it is saturated with respect to the transverse action of the system. In comparison, the next theorem we enounce, that we call "Arnol'd-Liouville with singularities", and which was proved by Zung in [Zun96], will be called a "semi-global" result, as we obtain a normal form of a *leaf* of the system.

3.1.2 Arnold-Liouville with singularities

One of the consequence of the existence of focus-focus and hyperbolic critical points is that there is a distinction between the leaf containing a point and the orbit through that point. The following proposition describes precisely how non-degenerate critical leaves are stratified by orbits of different Williamson types:

Proposition 3.1.4. *Let m be a point of Williamson type $\Bbbk$ of a proper, non-degenerate integrable system F. Then:*

1. *$\mathcal{O}_F(m)$ is diffeomorphic to a direct product $\mathbb{T}^c \times \mathbb{R}^o$ (and $c + o = k_\mathrm{x}$).*
2. *For any point m' in the closure of $\mathcal{O}_F(m)$, $k_\mathrm{e}(m') = k_\mathrm{e}$, $k_\mathrm{h}(m') \leqslant k_\mathrm{h}$ and $k_\mathrm{f}(m') \leqslant k_\mathrm{f}$.*
3. *The quantities k_e, $k_\mathrm{f} + c$ and $k_\mathrm{f} + k_\mathrm{h} + o$ are invariants of the leaf.*
4. *For a non-degenerate proper integrable system, a leaf Λ contains a finite number of F-orbits with a minimal $\Bbbk$ for $\preccurlyeq$, and the Williamson type for these leaves is the same.*

All these assertions are proven by Zung in [Zun96]. The last statement asserts that, in a non-degenerate critical leaf, a point with minimal Williamson type is not unique in general, but the minimal Williamson type is. This allows us to give the following definition.

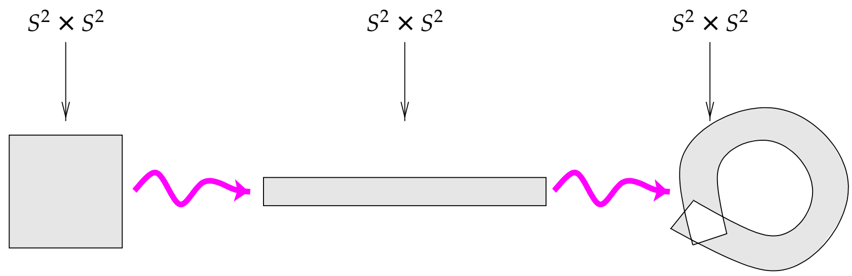

Figure 3.1: An image of a moment map overlapping itself.

Definition 3.1.5. *For a non-degenerate integrable system with b an element of the base space $\mathcal{B}$ of the foliation $\mathcal{F}$, the Williamson type of a leaf Λ_b is defined as*

$$\Bbbk(\Lambda_b) := \min_{\preccurlyeq}\{\Bbbk(m) \mid m \in \Lambda_b\} \ .$$

Let's mention here that although we have defined a notion of Williamson type for non-degenerate orbits and leaves, there is no relevant notion of a Williamson type for fibers. A simple (counter-)example to back up our remark is the following:

Example 3.1.6. *Let's take the symplectic 4-manifold $(M, \omega) = (\mathbb{S}^2_{r=1} \times \mathbb{S}^2_{r=2}, d\theta_1 \wedge dh_1 + d\theta_2 \wedge dh_2)$, and on it, the integrable system given by $F = (h_1, h_2)$. The image of its moment map is a simple 1×2-rectangular, that we will call $\mathbf{R}$.*

Now, we know that we can strech $\mathbf{R}$ to obtain a new figure $\mathbf{R}'$ corresponding to a $\mathbf{R}$ that overlaps itself (see Figure 3.1). In order to do so, we make use of a diffeomorphism S of $\mathbb{R}^2$ that produces a new moment map $S \circ F$, to which corresponds $\mathbf{R}'$.

On this image (Fig.3.1) we can see that certain edges and even vertices of the original rectangle $\mathbf{R} = F(M)$, and corresponding to $E - X$ or $E - E$ critical values are overlapping $\mathbf{R}$ at regular values. However, in the image of the moment map, this difference is not visible because the values overlap. In order to see it, we must investigate the base space of our system, or directly the manifold M. If we look at M for instance, the meaning of it is that the fiber of the edges and the vertices are not connected: different leaves are sent by $S \circ F$ to the same value.

In particular, one of these leaves corresponds to a regular Lagrangian 2-torus while another one corresponds to a — Lagrangian — fixed point. Hence, they don't have the same Williamson type, but they belong to the same fiber *by definition. This proves the desired result.*

Now, we define another notion that plays an important role in the Theorem we want to introduce.

Definition 3.1.7. *A non-degenerate critical leaf Λ is called* topologically stable *if there exists a saturated neighborhood $\mathcal{V}$ of Λ and a small neighborhood $\mathcal{U} \subset \mathcal{V}(\Lambda)$ of a point m of minimal rank, such that*

$$\forall \Bbbk \in \mathcal{W}_0^n \ , \ F(\mathrm{CrP}_{\Bbbk}(\mathcal{V}(\Lambda))) = F(\mathrm{CrP}_{\Bbbk}(\mathcal{U})) \ .$$

An integrable system will be called topologically stable *if all its critical points are non-degenerate and topologically stable.*

We understand the assumption of topological stability as a way to rule out some pathological behaviours that can occur *a priori* for general Lagrangian foliations. Note however that for all known examples, the non-degenerate critical leaves are all topologically stable, and it is conjectured that it is also the case for all analytic systems.

Since the papers [Zun96] and [Zun03] of Zung, the terminology concerning the assumption of topological stability has evolved. One speaks now of the *transversality assumption*, or of the *non-splitting condition*. This terminology was proposed first by Bolsinov and Fomenko in [BF04].

Yet, the expression "topological stability" was first justified because of the following proposition

Proposition 3.1.8 (Zung, [Zun96])**.** *Let F be an integrable Hamiltonian system, Λ a non-degenerate topologically stable critical leaf of rank k_{x}, and $\mathcal{U}(\Lambda)$ a saturated neighborhood of it. Then all critical leaves of rank k_{x} are topologically equivalent, all closed orbits of the Poisson action of $\mathbb{R}^n$ given by the moment map have the same dimension, and all critical sub-leaves are topologically stable.*

Definition 3.1.9. *A singularity of a proper integrable Hamiltonian system is (a germ of) a singular foliation: $(\mathcal{U}(\Lambda), \mathcal{F})$, where $\mathcal{U}(\Lambda)$ is a tubular neighborhood of the critical leaf. We say that two singularities are isomorphic if they are leaf-wise homeomorphic. We name the following singularities isomorphism classes "simple":*

- *A singularity is called of* (simple) elliptic type *if it is isomorphic to $\mathcal{L}^{\mathrm{e}}$: a plane $\mathbb{R}^2$ foliated by q_{e}.*
- *A singularity is called of* (simple) hyperbolic type *if it is isomorphic to $\mathcal{L}^{\mathrm{h}}$ with* **one** *hyperbolic critical point, where $\mathcal{L}^{\mathrm{h}}$ is given by a plane $\mathbb{R}^2$ locally foliated around the hyperbolic point by q_{h}. Since it comes from a proper integrable system, the leaves must be compact, so a connected component of the unstable manifold is linked to a connected component of the stable manifolds by homoclinic orbits. The two homoclinic orbits and the hyperbolic fixed point give the complete orbit-type stratification of the leaf. Thus, the critical leaf is homeomorphic to a "8", and the other regular leaves are homeomorphic to circles.*

- *A singularity is called of* (simple) focus-focus type *if it is isomorphic to* $\mathcal{L}^{\mathrm{f}}$*, where* $\mathcal{L}^{\mathrm{f}}$ *is given by* $\mathbb{R}^4$ *locally foliated by* q_1 *and* q_2*. One can show (see Proposition* 6.2 *in [VN00]) that the focus-focus critical leaf must be homeomorphic to a pinched torus* $\check{\mathbb{T}}^2$*: a 2-sphere with two points identified. The regular leaves around are regular tori. We describe in more details this singularity in Chapter 5.*

Properties of elliptic, hyperbolic and focus-focus singularities are discussed in details in [Zun96]. In particular, the fact that we can extend the Hamiltonian S^1-action that exists near a focus-focus point to a tubular neighborhood of the focus-focus singularity guarantees that the focus-focus critical fiber is indeed a pinched torus.

Assumption 3.1.10. *From now on, we will assume that all the systems we consider have* **simple topologically stable singularities***. In particular, simplicity implies that for the semi-toric systems we consider, focus-focus leaves will only have one vanishing cycle.*

Statement of the theorem

Now we can formulate the extension of Liouville-Arnold-Mineur theorem to singular leaves.

Theorem 3.1.11 (Arnold-Liouville with singularities, [Zun96])**.** *For* F *be a proper integrable system, let* Λ *be a non-degenerate critical leaf of Williamson type* $\Bbbk$ *and* $\mathcal{V}(\Lambda)$ *a saturated neighborhood of* Λ *with respect to* F*.*

Then the following statements are true:

1. *There exists an effective Hamiltonian action of* $\mathbb{T}^{k_{\mathrm{e}}+k_{\mathrm{f}}+k_{\mathrm{x}}}$ *on* $\mathcal{V}(\Lambda)$*. There is a locally free* $\mathbb{T}^{k_{\mathrm{x}}}$*-subaction. The number* $k_{\mathrm{e}}+k_{\mathrm{f}}+k_{\mathrm{x}}$ *is the maximal possible.*
2. *If* Λ *is topologically stable,* $(\mathcal{V}(\Lambda), \mathcal{F})$ *is leaf-wise diffeomorphic to an almost-direct product of elliptic, hyperbolic and focus-focus elementary singularities:* $(\mathcal{V}(\Lambda), \mathcal{F}) \simeq$

   $$\left((\mathcal{U}(\mathbb{T}^{k_{\mathrm{x}}}), \mathcal{F}_r) \times \mathcal{L}_1^{\mathrm{e}} \times \cdots \times \mathcal{L}_{k_{\mathrm{e}}}^{\mathrm{e}} \times \mathcal{L}_1^{\mathrm{h}} \times \cdots \times \mathcal{L}_{k_{\mathrm{h}}}^{\mathrm{h}} \times \mathcal{L}_1^{\mathrm{f}} \times \cdots \times \mathcal{L}_{k_{\mathrm{f}}}^{\mathrm{f}}\right)/\Gamma_0$$

   where $(\mathcal{U}(\mathbb{T}^{k_{\mathrm{x}}}), \mathcal{F}_r)$ *is a regular foliation by tori of a saturated neighborhood of* $\mathbb{T}^{k_{\mathrm{x}}}$ *and* Γ_0 *is a finite group that acts component-wise on the product (i.e: it commutes with projections on each component) and trivially on the elliptic components.*
3. *There exists partial action-angles coordinates on* $\mathcal{V}(\Lambda)$*: there exists a diffeomorphism* φ *such that*

   $$\varphi^*\omega = \sum_{i=1}^{k_{\mathrm{x}}} d\theta_i \wedge dI_i + P^*\omega_1$$

where $(\boldsymbol{\theta}, \mathbf{I})$ are the action-angles coordinates on $T^*\mathcal{T}$, where $\mathcal{T}$ is the zero torus in Miranda-Zung equivariant Normal form theorem stated in [MZ04] and ω_1 is a symplectic form on $\mathbb{R}^{n-k_{\mathrm{x}}} \simeq \mathbb{R}^{k_{\mathrm{e}}+k_{\mathrm{h}}+2k_{\mathrm{f}}}$.

Theorem 3.1.11 says in particular that under this mild assumption that is topological stability of the leaves, a critical leaf Λ is diffeomorphic to a (Γ_0-twisted) product of the simplest elliptic, hyperbolic and focus-focus leaves. The group Γ_0 describes how the hyperbolic flows, shall they come from a hyperbolic singularity or as the "hyperbolic" flow of a focus-focus singularity, twists the leaf. Again, it is trivial for a semi-toric system.

Remark 3.1.12. *It is very important to note that the assertion* 3. *of Theorem 3.1.11 explains that* **we don't have a symplectomorphism** *in the assertion* 2. *This is the "raison d'être" of all the work done in the topic, including this thesis. We will illustrate this fact in Section 3.3 by constructing action-angles coordinates around a focus-focus critical leaf, and show that we can regularize them so that they can be defined on the focus-focus critical leaf, but then we loose the fact that they are action-angle coordinates.*

Also, one should notice that in Theorem 3.1.11, it is only because we made the Assumption 3.1.10 that the singularity is leaf-wise **diffeomorphic** *to an almost-direct product of simple sigularities, and not only homeomorphic to it. Were there more than one pinch on the singuarity, we could only guarantee the existence of the homeomorphism.*

3.1.3 Williamson type stratification of the manifold

Definition 3.1.13. *For U an open subset of $F(M)$, we define the sheaf of critical values of a given Williamson type:*

$$\mathrm{CrV}_{\Bbbk}(U) = \{c \in U \mid \exists \Lambda \text{ a leaf of Williamson type } \Bbbk \text{ such that } \Lambda \subseteq F^{-1}(c)\}.$$

For V an open subset of $\mathcal{B}$, we can also define the sheaf of critical base points of the foliation of a given Williamson type:

$$\mathrm{CrL}_{\Bbbk}(V) = \{b \in V \mid \Lambda_b \text{ is of Williamson type } \Bbbk\}.$$

As a consequence of Theorem 3.1.11, we give this extension of Example 1.3.18, Corollaries 1.3.20 and 1.3.21 to the case of a integrable Hamiltonian action of a non-compact group, with a finer decomposing set:

Theorem 3.1.14. *Let (M, ω, F) be an integrable system. The mapping*

$$\begin{aligned} \mathbf{CrP(F)} : \mathcal{W}_0^n(F) &\to \{\mathrm{CrP}_{\Bbbk}^F(M) \mid \Bbbk \in \mathcal{W}_0(F)\} \\ \Bbbk &\mapsto \mathrm{CrP}_{\Bbbk}^F(M) \end{aligned}$$

stratifies M by symplectic submanifolds: $(M, \mathcal{C}^\infty(M \to \mathbb{R}), \mathbf{CrP(F)})$ is a symplectic stradispace.

Note that in Definition 1.3.25 the total space need not be a symplectic manifold (it doesn't even need to be a manifold), but here incidentally, (M, ω) is also a symplectic manifold.

Proof of Theorem. 3.1.14 With Eliasson normal form of a moment map on a small neighborhood $\mathcal{U}$ of a critical point m of Williamson type $\Bbbk$, we know that the set defined by the equations:

$$\begin{cases} x_1^{\mathrm{f}} = \xi_1^{\mathrm{f}} = \ldots = x_{2k_{\mathrm{f}}}^{\mathrm{f}} = \xi_{2k_{\mathrm{f}}}^{\mathrm{f}} = 0 \\ x_1^{h} = \xi_1^{h} = \ldots = x_{k_{\mathrm{h}}}^{h} = \xi_{k_{\mathrm{h}}}^{h} = 0 \\ x_1^{\mathrm{e}} = \xi_1^{\mathrm{e}} = \ldots = x_{k_{\mathrm{e}}}^{\mathrm{e}} = \xi_{k_{\mathrm{e}}}^{\mathrm{e}} = 0 \end{cases}$$

is the $\mathrm{CrP}_{\Bbbk}(\mathcal{U})$ in local coordinates. It is diffeomorkphic to an open set of $\mathbb{R}^{k_{\mathrm{x}}(m)}$: it is a $k_{\mathrm{x}}(m)$-dimensional manifold. Theorem 3.1.2 even tells us that on an orbital neighborhood $\tilde{\mathcal{U}}_m$ of m, $\mathrm{CrP}_{\Bbbk}(\tilde{\mathcal{U}}_m)$ is diffeomorphic to $\mathbb{T}^{k_{\mathrm{x}}} \times \mathbb{D}^{k_{\mathrm{x}}}$, and the restriction of the symplectic form to it is $\sum_{i=1}^{k_{\mathrm{x}}} d\theta_i \wedge dI_i$: it is again a symplectic form on $\mathrm{CrP}_{\Bbbk}(\tilde{\mathcal{U}}_m)$. Hence, since M is compact $\mathrm{CrP}_{\Bbbk}(M)$ is a symplectic manifold.

If we now take a point p near m such that $Q_{\Bbbk}(p)$ has now (apart from the I_i's) only one component $\neq 0$ and the other still $= 0$, we get that $p \in \mathrm{CrP}_{\Bbbk'}(\mathcal{U}_m)$, with a $\Bbbk' = (k_{\mathrm{e}} - \delta_{\mathrm{e}}, k_{\mathrm{h}} - \delta_{\mathrm{h}}, k_{\mathrm{f}} - \delta_{\mathrm{f}}, k'_{\mathrm{x}})$ with $\delta_{\mathrm{e,h,f}} = 0$ or 1 depending whether the non-zero component was part of an elliptic, hyperbolic or focus-focus block. This is a $\Bbbk'$ parent to $\Bbbk$, and we have $\mathrm{CrP}_{\Bbbk}(\mathcal{U}) \leq \mathrm{CrP}_{\Bbbk'}(\mathcal{U})$. By an immediate finite induction, the $\mathrm{CrP}_{\Bbbk}(\mathcal{U})$'s verify the frontier condition: they form a $\mathcal{W}_0(F)$-decomposition. The sheaf of functions on M is just the Poisson algebra $(\mathcal{C}^\infty(M \to \mathbb{R}), \{\cdot, \cdot\})$.

Now for the splitting condition, with Item 2. of Theorem 3.1.11 we see that we only need to treat the elliptic, hyperbolic and focus-focus cases with local models. In the $2n = 2$ elliptic and hyperbolic cases respectively, the $\mathrm{CrP}_E(\mathbb{R}^2)$ and $\mathrm{CrP}_H(\mathbb{R}^2)$ are just points: a neighborhood of the critical point is a disk, it is homeomorphic then isomorphic to the critical point times a cone over a small circle. For the focus-focus case, it is not more complicated: the $\mathrm{CrP}_{FF}(\mathbb{R}^4)$ is again a point, and we need to show there exists a 3-dimensional stradispace L such that a neighborhood of the focus-focus point is homeomorphic to this point times the cone over L. We can just take the 3-sphere $\mathbb{S}^3$ and take the cone over it: it is homeomorphic to the 4-ball, and hence is a neighborhood of a focus-focus point. □

Note that here the splitting condition was easy to prove as the whole space is a manifold, but it can be more complicated in general.

3.2 Critical values of a semi-toric integrable system

Eliasson's normal form theorem 1.2.23 and Miranda-Zung's theorem 3.1.2 give us a general understanding of the commutant of an almost-toric system: we have a local model of the image of the moment map near a critical value, that can be precised in the semi-toric case. We remind first these two lemmas.

Lemma 3.2.1. *Let $\mathcal{F}$ be a singular Liouville foliation given by a momentum map $F : M \to \mathbb{R}^k$. Let $\mathcal{F}'$ be a singular Liouville foliation given by a momentum map $F' : N \to \mathbb{R}^k$. If the level sets are locally connected, then for every smooth symplectomorphism*

$$\varphi : \mathcal{U} \subset M \to \mathcal{V} \subset N$$

where $\mathcal{U}$ is an open neighborhood of $p \in M$, $\mathcal{V}$ a neighborhood of $p' = \varphi(p) \in N$, and such that

$$\varphi^* \mathcal{F} = \mathcal{F}',$$

there exists a unique local diffeomorphism

$$G : (\mathbb{R}^k, F(p)) \to (\mathbb{R}^k, F'(p'))$$

such that

$$F \circ \varphi = G \circ F'.$$

Lemma 3.2.2. *Let (M, ω, F) be a proper, non-degenerate almost-toric integrable system. Then its fibers have a finite number of connected components.*

Proof of Lemma. 3.2.2: Let c be a value of F, and L be a connected component of $F^{-1}(c)$. On each point of L we can apply Eliasson's normal form. Since in an almost-toric integrable system the leaves are locally connected, this gives us the existence of an open neighborhood $\mathcal{V}(L)$ of L in which there is no other connected component of $F^{-1}(c)$.

Now, we have that $\bigcup_L \mathcal{V}(L)$ is an open covering of $F^{-1}(c)$, which is compact by the properness of F. We can thus extract a finite sub-covering of it. It implies that there is only a finite number of connected components. □

3.2.1 Symplectomorphisms preserving a semi-toric foliation

We deal here with the linear system $(L_{\Bbbk}, \omega_{\Bbbk}, Q_{\Bbbk})$, in the case $k_{\mathrm{h}} = 0$ and $k_{\mathrm{f}} = 1$. We rewrite

$$Q_{\Bbbk} = (q_1, q_2, q_{\mathrm{e}}^{(1)}, \ldots, q_{\mathrm{e}}^{(k_{\mathrm{e}})}, I_1, \ldots, I_{k_{\mathrm{x}}})$$

The theorem presented here gives precisions about the form of G.

Theorem 3.2.3. *Let F be a semi-toric integrable system, and m be a critical point of Williamson type $\Bbbk$, with $k_{\mathrm{f}} = 1$. Let $\varphi : V \subset (L_{\Bbbk}, \omega_{\Bbbk}) \to (\mathcal{O}^X(\mathcal{U}_m), \omega)$ be a symplectomorphism sending the transverse orbit of m on the zero-torus and such that*

$$F \circ \varphi \sim Q_{\Bbbk}.$$

Then there exists a unique diffeomorphism $G : V \subseteq \mathbb{R}^n \to U \subset F(M)$ such that:

1. *$F \circ \varphi = G \circ Q_{\Bbbk}$,*
2. *$\check{F} = A \cdot \check{Q}_{\Bbbk} + \check{F}(c)$, and with $A \in GL_{n-1}(\mathbb{Z})$.*

That is, the Jacobian of G is of the form:

$$\begin{pmatrix}
\partial_{q_1}G_1 & \partial_{q_2}G_1 & \partial_{q_{\mathrm{e}}^{(1)}}G_1 & \dots & \partial_{q_{\mathrm{e}}^{(k_{\mathrm{e}})}}G_1 & \partial_{I_1}G_1 & \dots & \partial_{I_{k_{\mathrm{x}}}}G_1 \\
0 & F^{\mathrm{f}} & F^{\mathrm{e}}_1 & \dots & F^{\mathrm{e}}_{k_{\mathrm{e}}} & F^{\mathrm{x}}_1 & \dots & F^{\mathrm{x}}_{k_{\mathrm{x}}} \\
0 & E^{\mathrm{f}}_1 & \ulcorner & & \urcorner & \ulcorner & & \urcorner \\
\vdots & \vdots & & E_{\mathrm{e}} & & & E_{\mathrm{x}} & \\
\vdots & E^{\mathrm{f}}_{k_{\mathrm{e}}} & \llcorner & & \lrcorner & \llcorner & & \lrcorner \\
\vdots & X^{\mathrm{f}}_1 & \ulcorner & & \urcorner & \ulcorner & & \urcorner \\
\vdots & \vdots & & X_{\mathrm{e}} & & & X_{\mathrm{x}} & \\
0 & X^{\mathrm{f}}_{k_{\mathrm{x}}} & \llcorner & & \lrcorner & \llcorner & & \lrcorner
\end{pmatrix} \qquad (\spadesuit)$$

Or, put in another way,

$$A = \begin{pmatrix} F^{\mathrm{f}} & F^{\mathrm{e}} & F^{\mathrm{x}} \\ E^{\mathrm{f}} & E^{\mathrm{e}} & E^{\mathrm{x}} \\ X^{\mathrm{f}} & X^{\mathrm{e}} & X^{\mathrm{x}} \end{pmatrix}$$

with:

- *$F^{\mathrm{f}} \in \mathbb{Z}$, $F^{\mathrm{e}} \in M_{1,k_{\mathrm{e}}}(\mathbb{Z})$, $F^{\mathrm{x}} \in M_{1,k_{\mathrm{x}}}(\mathbb{Z})$,*
- *$E^{\mathrm{f}} \in M_{k_{\mathrm{e}},1}(\mathbb{Z})$, $E^{\mathrm{e}} \in M_{k_{\mathrm{e}}}(\mathbb{Z})$, $E^{\mathrm{x}} \in M_{k_{\mathrm{x}}}(\mathbb{Z})$,*
- *$X^{\mathrm{f}} \in M_{k_{\mathrm{x}},1}(\mathbb{Z})$, $X^{\mathrm{e}} \in M_{k_{\mathrm{e}}}(\mathbb{Z})$, $X^{\mathrm{x}} \in M_{k_{\mathrm{x}}}(\mathbb{Z})$.*

This first theorem includes the particular case where $F = Q_{\Bbbk}$, that is, when we consider symplectomorphism that start and end with $Q_{\Bbbk}$. The existence of at least one symplectomorphism φ verifying the hypotheses of the theorem is a direct consequence of Miranda-Zung's Theorem 3.1.2 and the local connectedness of the fibers. The point of this theorem is to precise the form of G.

We can concatenate Miranda-Zung theorem and the statement above by saying that, given a semi-toric system F and a critical point with $k_{\mathrm{f}} = 1$, there exists an transverse-orbital neighborhood $\mathcal{U}(p)$ and a $A \in GL_{n-1}(\mathbb{Z})$ such that

$$F|_{\mathcal{U}(p)} \sim_{ST} (q_1, A \circ \check{Q}_{\Bbbk}).$$

Proof of Theorem. 3.2.3

Since φ is a symplectomorphism, it preserves the dynamics induced by the Hamiltonian vector fields of $\check{F}$. We have assumed that the components of $\check{F}$ have 2π-periodic flows. So, once pushed forward by φ, the vector fields must remain 2π-periodic. We have the expression

$$\chi_{f_i\circ\varphi^{-1}} = \partial_{q_1} G_i \chi_{q_1} + \partial_{q_2} G_i \chi_{q_2} + \sum_{j=1}^{k_{\mathrm{e}}} \partial_{q^{\mathrm{e}}_{(j)}} G_i \chi_{q^{\mathrm{e}}_{(j)}} + \sum_{j=1}^{k_{\mathrm{x}}} \partial_{I_j} G_i \chi_{I_j}.$$

The partial derivatives of G are constant under the action by the Hamiltonian flow of $Q_{\Bbbk}$.

Type of block	Critical Levelsets	Expression of the flow in local coordinates
Elliptic	$\{q_e = 0\}$ is a point in $\mathbb{R}^2$	$\phi^t_{q_e} : z^{\mathrm{e}} \mapsto e^{it} z^{\mathrm{e}}$
Hyperbolic	$\{q_h = 0\}$ is the union of the lines $\{x^{\mathrm{h}} = 0\}$ and $\{\xi^{\mathrm{h}} = 0\}$ in $\mathbb{R}^2$	$\phi^t_{q_h} : (x^{\mathrm{h}}, \xi^{\mathrm{h}}) \mapsto (e^{-t} x^{\mathrm{h}}, e^t \xi^{\mathrm{h}})$
Focus-focus	$\{q_1 = q_2 = 0\}$ is the union of the planes $\{x_1 = x_2 = 0\}$ and $\{\xi_1 = \xi_2 = 0\}$ in $\mathbb{R}^4$	$\phi^t_{q_1} : (z^{\mathrm{f}}_1, z^{\mathrm{f}}_2) \mapsto (e^{-t} z^{\mathrm{f}}_1, e^t z^{\mathrm{f}}_2)$ and $\phi^t_{q_2} : (z^{\mathrm{f}}_1, z^{\mathrm{f}}_2) \mapsto (e^{it} z^{\mathrm{f}}_1, e^{it} z^{\mathrm{f}}_2)$

Table 3.1: Properties of elementary blocks

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The formula of the flow given in the Tab.3.2.1 give us an explicit expression of the flow :

$$\phi^t_{f_i\circ\varphi^{-1}}\left(z_1, z_2, x^{\mathrm{e}}_1, \xi^{\mathrm{e}}_1, \ldots, x^{\mathrm{e}}_{k_{\mathrm{e}}}, \xi^{\mathrm{e}}_{k_{\mathrm{e}}}, \theta_1, I_1, \ldots, \theta_{k_{\mathrm{x}}}, I_{k_{\mathrm{x}}}\right) =$$
$$\Big(e^{(\partial_{q_1} G_i + i\partial_{q_2} G_i)t} z_1, e^{(-\partial_{q_1} G_i + i\partial_{q_2} G_i)t} z_2, z^{\mathrm{e}}_1 e^{i\partial_{q^{\mathrm{e}}_{(1)}} G_i \cdot t}, \ldots, z^{\mathrm{e}}_{k_{\mathrm{e}}} e^{i\partial_{q^{\mathrm{e}}_{(k_{\mathrm{e}})}} G_i \cdot t},$$
$$\theta_1 + \partial_{I_1} G_i \cdot t, \ldots, \theta_{k_{\mathrm{x}}} + \partial_{I_{k_{\mathrm{x}}}} G_i \cdot t, I_1, \ldots, I_{k_{\mathrm{x}}}\Big).$$

So necessarily, for $i = 2, \ldots, n$ and $j = 1, \ldots, n$, we have first $\partial_{q_1} G_i = 0$, and we also have

$$\partial_{q_2} G_i, \partial_{q^{\mathrm{e}}_{(j)}} G_i \in \mathbb{Z} \text{ and } \partial_{I_{k_{\mathrm{x}}}} G_i \in \mathbb{Z}.$$

If a coefficient of the Jacobian is integer on $\varphi(\mathcal{U})$, it must be constant on it. This shows that $\check{F} = A \circ \check{Q}$ with $A \in M_{n-1}(\mathbb{Z})$. Now, A is invertible because G is a local diffeomorphism, and since the components of $\check{F}$ are 2π-periodic, we have that necessarily $A^{-1} \in M_{n-1}(\mathbb{Z})$.

□

Note that this result here only uses the 2π-periodicity of the flow, and no other assumption about the dynamics of F. In the next theorem, we have the *same* foliation before and after composing with φ. This stronger statement will get us precisions about the form of $Jac(G)$, in particular the unicity of its infinite jet on the set of critical values.

3.2.2 Transition functions between the semi-toric local models

I - Symplectomorphisms preserving a linear semi-toric foliation

In this section, we need to precise our notion of flat function. To this end, let's introduce the following set:

Definition 3.2.4. *Let S be a subset of $\subseteq \mathbb{R}^k$. We define the set $\mathcal{F}\ell^{\infty}_S(\mathcal{U})$ as the set of real-valued smooth functions on $\mathcal{U} \subseteq \mathbb{R}^k$ which are flat in any direction for all the points $x \in S$.*

We recall the bold notations for row vectors $\mathbf{x} = (x_1, \dots, x_r)$ (e.g.: $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{k_\mathrm{x}})$, $\mathbf{I} = (I_1, \dots, I_{k_\mathrm{x}})$). We also fix the convention, for row vectors $\mathbf{x}$ and $\mathbf{y}$ of the same size and z a single coordinate:

$$d\mathbf{x} \wedge d\mathbf{y} = \sum_{j=1}^{r} dx_j \wedge dy_j \quad \text{and} \quad d\mathbf{x} \wedge dz = \sum_{j=1}^{r} dx_j \wedge dz \tag{3.1}$$

What we prove here is a uniqueness theorem for the G of whom we proved the existence in Theorem 3.2.3. For this reason, we shall call the diffeomorphism B here, because the uniqueness of G in that case is an information about the change of *Basis* that occurs.

Theorem 3.2.5. *Let $(L_\Bbbk, \omega_\Bbbk, Q_\Bbbk)$ be a linear model with $k_\mathrm{f} = 1$. Let ψ be a symplectomorphism of $\mathcal{U} \subset L_\Bbbk$ an open neighborhood of the orbit of Williamson type $\Bbbk$, which preserves the foliation $\mathcal{Q}_\Bbbk$.*

Then the diffeomorphism $B : V \to U$ is such that there exists $\epsilon^\mathrm{f}_1, \epsilon^\mathrm{f}_2 \in \{-1, +1\}$, a matrix $\boldsymbol{\epsilon}^\mathrm{e} \in Diag_{ke}(\{-1, +1\})$ and a function $u \in \mathcal{F}\ell^{\infty}_S(\mathcal{U})$ where $S = \{q_1 = 0, q_2 = 0\} \subset V$ so that in $Jac(B)$ we have:

1. *$B_1(Q_\Bbbk) = \epsilon^\mathrm{f}_1 q_1 + u$,*
2. *$E^\mathrm{f} = 0$, $E^\mathrm{e} = \boldsymbol{\epsilon}^\mathrm{e}$ and $E^\mathrm{x} = 0$,*

3. $X^{\mathrm{x}} \in GL_{k_{\mathrm{x}}}(\mathbb{Z})$*,*

4. $F^{\mathrm{f}} = \epsilon_2^{\mathrm{f}}$*,* $F^{\mathrm{e}} = 0$ *and* $F^{\mathrm{x}} = 0$*.*

That is, we have, for $\tilde{x} = x \circ \psi^{-1}$*:*

a. $\tilde{q}_1 = \epsilon_1^{\mathrm{f}} q_1 + u$ *,* $\tilde{q}_2 = \epsilon_2^{\mathrm{f}} q_2$*,*

b. $\widetilde{q_{\mathrm{e}}^{(i)}} = \epsilon_i^{\mathrm{e}} q_{\mathrm{e}}^{(i)}$*,*

c. $(\tilde{I}_1, \ldots, \tilde{I}_{k_{\mathrm{x}}}) = (X^{\mathrm{f}} | X^{\mathrm{e}} | X^{\mathrm{x}}) \circ \check{Q}_{\Bbbk}$*.*

For the Jacobian, it means that

$$Jac(B) = \begin{pmatrix} \epsilon_1^{\mathrm{f}} + u_1 & u_2 & \ldots & \ldots & \ldots & \ldots & \ldots & u_n \\ 0 & \epsilon_2^{\mathrm{f}} & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & \epsilon_1^{\mathrm{e}} & \ddots & \vdots & \vdots & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \epsilon_{k_{\mathrm{e}}}^{\mathrm{e}} & 0 & \ldots & 0 \\ \vdots & X_1^{\mathrm{f}} & \ulcorner & & \urcorner & \ulcorner & & \urcorner \\ \vdots & \vdots & & X^{\mathrm{e}} & & & X^{\mathrm{x}} & \\ 0 & X_{k_{\mathrm{x}}}^{\mathrm{f}} & \llcorner & & \lrcorner & \llcorner & & \lrcorner \end{pmatrix} (\bigstar)$$

with the u_i*'s being in* $\mathcal{F}\ell_S^{\infty}(\mathcal{U})$*.*

For practical uses, we are not interested in the precise form of u, but just by the fact that it is flat on S.

The theorem was proved in the focus-focus case for $2n = 4$ by San Vũ Ngọc in [VN03]. We follow the same ideas to give here a proof in the general case.

Proof of Theorem. 3.2.5

As a particular case of theorem 3.2.3, we already know that $Jac(B)$ is of the form (♠). The point here is to exploit the fact that the linear model $Q_{\Bbbk}$ has specific dynamical features conserved by a canonical transformation.

Points (1),(2) and (3):

A point fixed by the flow of a Hamiltonian H is preserved by a symplectomorphism: its image will be a fixed point for the precomposition of H by a symplectomorphism. Theorem 3.1.2 tells us how critical loci of a given Williamson type come as "intersections" of other critical loci. In particular, we know that for $i = 1, \ldots, k_{\mathrm{e}}$, there exists a $c_i \in V$ such that $X_{q_{\mathrm{e}}^{(i)}}(c_i) = 0$, $X_{q_{\mathrm{e}}^{(j)}}(c_i) \neq 0$ for $j \neq i$ and $X_{q_2}(c_i) \neq 0$ (the X_{I_i}'s never cancel). Now, we have the following formula

$$X_{q_{\mathrm{e}}^{(i)} \circ \psi^{-1}}(c_i) = E_i^{\mathrm{f}} X_{q_2}(c_i) + \sum_{j=1}^{k_{\mathrm{e}}} E_{ij}^{\mathrm{e}} X_{q_{\mathrm{e}}^{(j)}}(c_i) + \sum_{j=1}^{k_{\mathrm{f}}} E_{ij}^{\mathrm{x}} X_{I_j}(c_i) = 0.$$

The family

$$(X_{q_2}(c_i), X_{q_e^{(1)}}(c_i), \dots, X_{q_e^{(i-1)}}(c_i), X_{q_e^{(i+1)}}(c_i), \dots, X_{q_e^{(k_e)}}(c_i), X_{I_1}, \dots, X_{I_{k_x}})$$

is a free family, so we have for $j \neq i$ that $E^{\mathrm{e}}_{ij} = 0$, and $E^{\mathrm{x}}_{ij} = 0$ for all $j = 1, \dots, k_{\mathrm{x}}$. This is true for all $i = 1, \dots, k_{\mathrm{e}}$.

If we take a c in $\{q_1 = q_2 = 0\}$, we get by a similar reasoning that $F^{\mathrm{e}} = 0$ and $F^{\mathrm{x}} = 0$. Since $\det(A) = \pm 1$ we have necessary that $F^{\mathrm{f}} = \pm 1$ and $E^{\mathrm{e}}_{ij} = \pm 1$. Lastly, we have that $\det(X_{\mathrm{x}}) = \pm 1$.

The transverse fields have no fixed point, so we cannot say more concerning the coefficients of $X_{I_j \circ \psi^{-1}}$ than what we said in Theorem 3.2.3 without extra assumptions on the torus action.

Point (4), part 1:

Now, we need to deal with the flow of q_1, that is, the hyperbolic flow.

A leaf Λ of $\mathcal{Q}_{\Bbbk}$ of Wiliamson type $\Bbbk$ with $k_{\mathrm{f}} = 1$ is stable by the flow of q_1. On it, the flow is radial: for a point $m' \in \Lambda$ of Williamson type $\Bbbk' = (k_{\mathrm{e}}, 0, 0, k_{\mathrm{x}})$, there exists a unique point m on the zero-torus of the leaf Λ such that the segment $[m', m[$ is a trajectory for q_1. Depending on whether m' is on the stable $(+)$ or the unstable $(-)$ manifold, we have that $[m', m[= \{\phi^{\pm t}_{q_1}(m') \mid t \in [0, \infty[\}$. Remember that m is a fixed point for q_1.

We then have that $\psi([m', m[)$ must be a trajectory of $q_1 \circ \psi^{-1} = B_1 \circ Q_{\Bbbk}$. We also have the explicit expression for the flow of $q_1 \circ \psi^{-1}$

$$\chi_{q_1 \circ \psi^{-1}} = \partial_{q_1} B_1 \cdot \chi_{q_1} + \partial_{q_2} B_1 \cdot \chi_{q_2} + \sum_{j=1}^{k_{\mathrm{e}}} \partial_{q_e^{(j)}} B_1 \cdot \chi_{q_e^{(j)}} + \sum_{j=1}^{k_{\mathrm{x}}} \partial_{I_j} B_1 \cdot \chi_{I_j}.$$

Firstly, the limit of $\phi^t_{q_1 \circ \psi^{-1}}(\psi(m'))$ must be a fixed point of $q_1 \circ \psi^{-1}$. Since the χ_{I_i}'s have no fixed point, we necessarily have that

$$\forall c \in \mathrm{CrV}^{Q_{\Bbbk}}_{\Bbbk}(U) \text{ and } \forall j = 1, \dots, k_{\mathrm{x}}\ ,\ \partial_{I_j} B_1(c) = 0. \tag{3.2}$$

Since the result is true for all $\Bbbk$ with $k_{\mathrm{f}} = 1$, if $k_{\mathrm{e}} \geqslant 1$, we can apply the same reasoning and get equation 3.2 for $\mathrm{CrV}_{\tilde{\Bbbk}}(U)$ with $\tilde{\Bbbk} = (0, 0, k_{\mathrm{f}}, k_{\mathrm{x}} + k_{\mathrm{e}})$, that is

$$\forall \tilde{c} \in \mathrm{CrV}^{Q_{\Bbbk}}_{\tilde{\Bbbk}}(U) \text{ and } \forall j = 1, \dots, k_{\mathrm{e}}\ ,\ \partial_{q_e^{(j)}} B_1(\tilde{c}) = 0. \tag{3.3}$$

Since for these $\Bbbk$ and $\tilde{\Bbbk}$ we have $\mathrm{CrV}^{Q_{\Bbbk}}_{\Bbbk}(U) \subseteq \overline{\mathrm{CrV}^{Q_{\Bbbk}}_{\tilde{\Bbbk}}(U)}$, comes

$$\forall c \in \mathrm{CrV}^{Q_{\Bbbk}}_{\Bbbk}(U) \text{ and } \forall j = 1, \dots, k_{\mathrm{e}}\ ,\ \partial_{q_e^{(j)}} B_1(c) = 0. \tag{3.4}$$

Now that we know there is no transverse nor elliptic component in the flow of $q_1 \circ \psi^{-1}$ for critical leaves with focus-focus component, let's focus on the q_2-component. The image trajectory $\psi([m', m[)$ is contained in a 2-dimensional plane (the stable or unstable manifold). Since ψ is smooth at m', $\psi([m',m[)$ is even contained in a sector of this plane that also contains m'. Remembering that $\psi([m',m[)$ is a trajectory *for an infinite time*, the only linear combinations of χ_{q_1}, χ_{q_2} which yields trajectories such that that $(a_1'(t), a_2'(t))$ remains in a fixed sector are multiples of χ_{q_1}. So we have that

$$\forall c \in \mathrm{CrV}_{\Bbbk}^{Q_{\Bbbk}}(U)\ ,\ \partial_{q_2} B_1(c) = 0. \tag{3.5}$$

This shows actually that B_1 is flat in the variables $(q_2, \mathbf{q}_{\mathrm{e}}, \mathbf{I})$ on $\mathrm{CrV}_{\Bbbk}^{Q_{\Bbbk}}(U)$.

Point (4), part 2:

To show that $B_1 - q_1$ is flat on $\mathrm{CrV}_{\Bbbk}^{Q_{\Bbbk}}(U)$ in all the variables, we can now treat the variables $(q_{\mathrm{e}}^{(1)}, \ldots, q_{\mathrm{e}}^{(k_{\mathrm{e}})}, I_1, \ldots, I_{k_{\mathrm{x}}})$ as parameters. We can always suppose that ψ preserves the stable and unstable manifold of q_1: this assumption is equivalent to fix the sign of $\partial_1 B_1$ to be positive on $\mathcal{U}$. As a result we'll have $\epsilon_1^{\mathrm{f}} = 1$. And again, we can assume that $k_{\mathrm{e}} = 0$, as flatness is a closed property: here, it is stable when taking the limit $q_{\mathrm{e}}^{(i)} = 0$.

With the explicit expression of the flow of q_1 and q_2, if we set $\bar{z}_1 z_2 = c$ and $z_2 = \bar{\delta}$, we have for the joint flow of q_1 and q_2 at respective times $s = \ln\left|\frac{\delta}{c}\right|$ and $t = \arg(\delta) - \arg(c)$

$$\underbrace{\phi_{q_1}^{s} \circ \phi_{q_2}^{t}}_{=:\Upsilon}(c, \bar{\delta}, \boldsymbol{\theta}, \mathbf{I}) = (\delta, \bar{c}, \boldsymbol{\theta}, \mathbf{I}). \tag{3.6}$$

One can then state the fact that Υ is a smooth and single-valued function in a neighborhood containing $\{(0, \bar{\delta}, \boldsymbol{\theta}, \mathbf{I}), \boldsymbol{\theta} \in \mathbb{T}^{k_{\mathrm{x}}}, \mathbf{I} \in \mathcal{B}^{k_{\mathrm{x}}}(0,\eta)\}$. Now, we know that $\psi^{-1}(0, \bar{\delta}, \boldsymbol{\theta}, \mathbf{I})$ is of the form $(0, a, \boldsymbol{\theta}', \mathbf{I})$ and $\psi^{-1}(\delta, 0, \boldsymbol{\theta}, \mathbf{I})$ is of the form $(b, 0, \boldsymbol{\theta}'', \mathbf{I})$, since ψ preserve the level sets and the stable and unstable manifolds. Hence, for $\psi^{-1} \circ \Upsilon \circ \psi$

$$(0, a, \boldsymbol{\theta}', \mathbf{I}) \overset{\psi}{\mapsto} (0, \bar{\delta}, \boldsymbol{\theta}, \mathbf{I}) \overset{\Upsilon}{\mapsto} (\delta, 0, \boldsymbol{\theta}, \mathbf{I}) \overset{\psi^{-1}}{\mapsto} (b, 0, \boldsymbol{\theta}'', \mathbf{I}).$$

With the expression of Υ in (3.6), we know that in the complementary set of $\{z_1 = 0\}$, $\psi^{-1} \circ \Upsilon \circ \psi$ is equal to the joint flow of $B(q_1, q_2, \mathbf{I})$ at the multi-time $(\ln\left|\frac{\delta}{c}\right|, \arg(\delta) - \arg(c), 0, \ldots, 0)$. With what we already know about the flow of $q_2 \circ \psi^{-1}$, when we write the joint flow in terms of the flows of the components of $Q_{\Bbbk}$ we get:

$$\begin{aligned}
\phi_{B(q_1,q_2,\mathbf{I})}^{(\ln\left|\frac{\delta}{c}\right|, \arg(\delta)-\arg(c))} &= \phi_{\partial_1 B_1 \cdot q_1 + \partial_2 B_1 \cdot q_2 + \sum_j \partial_{I_j} B_1 \cdot I_j}^{\ln\left|\frac{\delta}{c}\right|} \circ \phi_{q_2}^{\arg(\delta)-\arg(c)} \\
&= \phi_{q_1}^{\partial_1 B_1 \cdot \ln\left|\frac{\delta}{c}\right|} \circ \phi_{q_2}^{\partial_2 B_1 \cdot \ln\left|\frac{\delta}{c}\right| + \arg(\delta) - \arg(c)} \\
&\qquad \circ \underbrace{\phi_{I_1}^{-\partial_{I_1} B_1 \cdot \ln\left|\frac{c}{\delta}\right|} \circ \ldots \circ \phi_{I_{k_{\mathrm{x}}}}^{-\partial_{I_{k_{\mathrm{x}}}} B_1 \cdot \ln\left|\frac{c}{\delta}\right|}}_{=id \text{ (c.f. Point 4.(1))}}.
\end{aligned}$$

Since $\psi^{-1} \circ \Upsilon \circ \psi$ is smooth at the origin, it's also smooth in a neighborhood of the origin; for c small enough, we can look at the first component of the flow on $(c, a, \boldsymbol{\theta}, \mathbf{I})$: here c shall be the variable while a, $\boldsymbol{\theta}$ and $\mathbf{I}$ are parameters. We have the application

$$\begin{aligned} c \mapsto & e^{\partial_1 B_1 \cdot \ln\left|\frac{\delta}{c}\right| + i(-\partial_2 B_1 . \ln\left|\frac{\delta}{c}\right| + \arg(\delta) - \arg(c))} c \\ & = \left[e^{\partial_1 B_1 \ln|\delta| + i(\partial_2 B_1 \ln|\delta| + \arg(\delta))}\right] e^{(1-\partial_1 B_1)\cdot \ln|c| + i(-\partial_2 B_1 \ln|c|)}. \end{aligned}$$

The terms in brackets are obviously a smooth function of c, and so the last exponential term is also smooth as a function of c on 0. Hence, the real part and the imaginary part are both smooth functions of (c_1, c_2) in $(0, 0, \mathbf{I})$.

We then have the following lemma:

Lemma 3.2.6. *Let $f \in \mathcal{C}^\infty(\mathbb{R}^k \to \mathbb{R})$ be a smooth function such that: $x \mapsto f(x)\ \ln \|x\|$ is also a smooth function.*

Then f is necessarily flat in 0 in the k variables $(x_1, \ldots, x_k)$.

With this elementary lemma, we have that $(1 - \partial_1 B_1) \circ \alpha$ and $\partial_2 B_1 \circ \alpha$ are flat for all $(0, 0, \mathbf{I})$, where $\alpha(c_1, c_2, \mathbf{I}) = (c_1\delta_1 + c_2\delta_2, c_1\delta_2 - c_2\delta_1, \mathbf{I})$. The function α is a linear function, and it is invertible since $\delta \neq 0$. This gives us the flatness of $1 - \partial_1 B_1$ and $\partial_2 B_1$, as functions of c_1 and c_2 for all the $(0, 0, \mathbf{I})$, and thus, as functions of all the n variables for all the $(0, 0, \mathbf{I})$.

Supposing that ψ exchanges stable and unstable manifolds yields the same demonstration *mutatis mutandis*, that is, in the last part of the proof, if we look at the first component of the flow on $(\bar{c}, a, \boldsymbol{\theta}, \boldsymbol{\xi})$ with a, $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$ understood as parameters. $\square$

In the end the functions are flat on a codimension-2 manifold.

In Theorem 3.2.3, we associate to a symplectomorphism that preserves a semi-toric foliation a unique G of the form (♠). It would be interesting to have more knowledge about the restrictions on the form of such symplectomorphism, with, for instance, further investigation in the direction of Theorem 2.1.4. However, in this thesis, we are more interested in describing the image of the moment map, so we'd like to know "how many differents" G can occur in Theorem 3.2.3.

Theorem 3.2.5 can be applied to answer to this question. If we think of the G's as "local models" of the image of the moment map, then Theorem 3.2.5 describes the "transition functions" B. Indeed, if we have two local models given by Theorem 3.2.3, on $\mathcal{U} \subseteq M$, a neighborhood of a point m of Williamson type $\Bbbk$, one has:

$$\begin{cases} F \circ \varphi = G \circ Q_{\Bbbk}, \\ F \circ \varphi' = G' \circ Q_{\Bbbk}. \end{cases}$$

Then we get:

$$Q_{\Bbbk} \circ \underbrace{(\varphi^{-1} \circ \varphi')}_{=\psi} = \underbrace{(G^{-1} \circ G')}_{=B} \circ Q_{\Bbbk}.$$

We can apply Theorem 3.2.5 to the pair ($\psi = \varphi^{-1} \circ \varphi', B = G^{-1} \circ G'$) and then get the relation:

$$G' = (\epsilon_1^{\mathrm{f}} G_1 + u, \epsilon_2^{\mathrm{f}} G_2, \epsilon_1^{\mathrm{e}} G_3, \ldots, \epsilon_{k_{\mathrm{e}}}^{\mathrm{e}} G_{k_{\mathrm{e}}+2}, (X^{\mathrm{f}}|X^{\mathrm{e}}|X^{\mathrm{x}}) \circ \check{G})\ ,\ u \in \mathcal{F}\ell_S^{\infty}(U). \tag{3.7}$$

Symplectic invariants for the transition functions

We have given some restriction on the transition function B between two local models of a semi-toric system. If we now authorize ourselves to change the symplectomorphisms in the local models, we can again have a "nicer" G. This amounts to determine what, in B, is a semi-local *symplectic* invariant of the system. Let's first set a notation: for $A, B \in M_{p,q}(R)$, $A \bullet B = (a_{ij} b_{ij})$, and

$$E_1 = \frac{1}{2} \begin{pmatrix} 1+\epsilon_1^{\mathrm{f}} & 1-\epsilon_1^{\mathrm{f}} & 0 & 0 \\ -1+\epsilon_1^{\mathrm{f}} & 1+\epsilon_1^{\mathrm{f}} & 0 & 0 \\ 0 & 0 & 1+\epsilon_1^{\mathrm{f}} & 1-\epsilon_1^{\mathrm{f}} \\ 0 & 0 & -1+\epsilon_1^{\mathrm{f}} & 1+\epsilon_1^{\mathrm{f}} \end{pmatrix}$$

$$E_2 = \frac{1}{2} \begin{pmatrix} 1+\epsilon_2^{\mathrm{f}} & 0 & 1-\epsilon_2^{\mathrm{f}} & 0 \\ 0 & 1+\epsilon_2^{\mathrm{f}} & 0 & 1-\epsilon_2^{\mathrm{f}} \\ 1-\epsilon_2^{\mathrm{f}} & 0 & 1+\epsilon_2^{\mathrm{f}} & 0 \\ 0 & 1-\epsilon_2^{\mathrm{f}} & 0 & 1+\epsilon_2^{\mathrm{f}} \end{pmatrix}.$$

Theorem 3.2.7. *Let's consider the diffeomorphisms*

$$\zeta_{X^{\mathrm{f}},X^{\mathrm{e}}}(z_1, z_2, \mathbf{x}^{\mathrm{e}}, \boldsymbol{\xi}^{\mathrm{e}}, \boldsymbol{\theta}, \mathbf{I}) = \\ (e^{-i\boldsymbol{\theta}\cdot X^{\mathrm{f}}} z_1, e^{-i\boldsymbol{\theta}\cdot X^{\mathrm{f}}} z_2, e^{-i\boldsymbol{\theta}\cdot (X^{\mathrm{e}})^t} \bullet \mathbf{z}^{\mathrm{e}}, \boldsymbol{\theta}, \mathbf{I} + Q_{\Bbbk}^{\mathrm{e}} \cdot (X^{\mathrm{e}})^t + q_2 \cdot (X^{\mathrm{f}})^t)$$

and

$$\eta_{\epsilon_1^{\mathrm{f}},\epsilon_2^{\mathrm{f}},X^{\mathrm{x}}}(x_1, \xi_1, x_2, \xi_2, \mathbf{x}^{\mathrm{e}}, \boldsymbol{\xi}^{\mathrm{e}}, \boldsymbol{\theta}, \mathbf{I}) = ((x_1, \xi_1, x_2, \xi_2) E^t_{\epsilon_1^{\mathrm{f}},\epsilon_2^{\mathrm{f}}}, \mathbf{z}^{\mathrm{e}}, \boldsymbol{\theta} \cdot (X^{\mathrm{x}})^{-1}, \mathbf{I} \cdot (X^{\mathrm{x}})^t)$$

where $E_{\epsilon_1^{\mathrm{f}},\epsilon_2^{\mathrm{f}}} = E_1 E_2$.

Then we have that $\zeta_{X^{\mathrm{f}},X^{\mathrm{e}}}$ *and* $\eta_{\epsilon_1^{\mathrm{f}},\epsilon_2^{\mathrm{f}},X^{\mathrm{x}}}$ *are symplectomorphisms of* $L_{\Bbbk}$ *which preserve the foliation* $\mathcal{Q}_{\Bbbk}$*. In particular, we have that, if* B *is a local diffeomorphism of* $\mathbb{R}^n$ *of the form* ($\bigstar$)*, then*

$$B \circ Q_{\Bbbk} = \varphi^* \left(\begin{pmatrix} id_2 & 0 & 0 \\ 0 & \boldsymbol{\epsilon}^{\mathrm{e}} & 0 \\ 0 & 0 & id_{k_{\mathrm{x}}} \end{pmatrix} \circ Q_{\Bbbk} \circ (\zeta_{X^{\mathrm{f}},X^{\mathrm{e}}} \circ \eta_{\epsilon_1^{\mathrm{f}},\epsilon_2^{\mathrm{f}},X^{\mathrm{x}}}) + (u, 0, \ldots, 0) \right) \tag{3.8}$$

with $u \in \mathcal{F}\ell^{\infty}(S)$.

This theorem means that as far as the linear part of the local model is concerned, its only symplectic invariants are the orientations of the half-spaces given by elliptic critical values.

Remark 3.2.8. *The presence of transpose matrices in the theorem comes only from the fact that, strictly speaking, $Q_{\Bbbk}$ is a row vector. Only for this theorem do we rely on this convention of writing.*

Proof of Theorem. 3.2.7
First let's prove that q_1 and q_2 are preserved:

$$\zeta^*_{X^{\mathrm{f}},X^{\mathrm{e}}}(q_1 + iq_2) = \zeta^*_{X^{\mathrm{f}},X^{\mathrm{e}}}(\bar{z}_1 z_2) = e^{i\boldsymbol{\theta}\cdot X^{\mathrm{f}}}\bar{z}_1 e^{-i\boldsymbol{\theta}\cdot X^{\mathrm{f}}} z_2 = \bar{z}_1 z_2 = q_1 + iq_2.$$

$$\zeta^*_{X^{\mathrm{f}},X^{\mathrm{e}}} Q^{\mathrm{e}}_{\Bbbk} = (|e^{-i\boldsymbol{\theta}\cdot(X^{\mathrm{e}}_{1,\cdot})^t}\cdot z^{\mathrm{e}}_1|^2, \dots, |e^{-i\boldsymbol{\theta}\cdot(X^{\mathrm{e}}_{k_{\mathrm{e}},\cdot})^t}\cdot z^{\mathrm{e}}_{k_{\mathrm{e}}}|^2) = (|z^{\mathrm{e}}_1|^2, \dots, |z^{\mathrm{e}}_{k_{\mathrm{e}}}|^2) = Q^{\mathrm{e}}_{\Bbbk}$$

$$\zeta^*_{X^{\mathrm{f}},X^{\mathrm{e}}}\mathbf{I} = q_2 \cdot X^{\mathrm{f}} + Q^{\mathrm{e}}_{\Bbbk}\cdot (X^{\mathrm{e}})^t + \mathbf{I}.$$

For η, we have: $\eta^* Q_{\Bbbk} = (E_1E_2)^* Q^{\mathrm{f}}_{\Bbbk} + (X^{\mathrm{x}})^* Q^{\mathrm{x}}_{\Bbbk}$, so we can treat each action separately. We can also treat E_1 and E_2 separately, as the two matrices commutes, and treat only the case when ϵ^{f}_1 (respectively ϵ^{f}_2) is equal to -1, for when $\epsilon^{\mathrm{f}}_i = +1$, $E_i = id$.

– $\epsilon^{\mathrm{f}}_1 = -1$:

$$E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \text{ so } E_1 \cdot \begin{pmatrix} x_1 \\ \xi_1 \\ x_2 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ -x_1 \\ \xi_2 \\ -x_2 \end{pmatrix} = \begin{pmatrix} \hat{x}_1 \\ \hat{\xi}_1 \\ \hat{x}_2 \\ \hat{\xi}_2 \end{pmatrix}$$

$$E_1^* q_1 = \hat{x}_1\hat{\xi}_1 + \hat{x}_2\hat{\xi}_2 = -\xi_1 x_1 - \xi_2 x_2 = -q_1 = \epsilon^{\mathrm{f}}_1 q_1,$$
$$E_1^* q_2 = \hat{x}_1\hat{\xi}_2 - \hat{x}_2\hat{\xi}_1 = \xi_1(-x_2) - \xi_2(-x_1) = q_2.$$

– $\epsilon^{\mathrm{f}}_2 = -1$:

$$E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$E_2^* q_1 = \tilde{x}_1\tilde{\xi}_1 + \tilde{x}_2\tilde{\xi}_2 = x_2\xi_2 + x_1\xi_1 = q_1,$$
$$E_2^* q_2 = \tilde{x}_1\tilde{\xi}_2 - \tilde{x}_2\tilde{\xi}_1 = x_2\xi_1 - x_1\xi_2 = -q_2 = \epsilon^{\mathrm{f}}_2 q_2.$$

And $(X^{\mathrm{x}})^* Q^{\mathrm{x}}_{\Bbbk} = \mathbf{I} \cdot (X^{\mathrm{x}})^t$.

What is left is to prove the preservation of ω. For ζ we have:

$$\zeta^*_{X^{\mathrm{f}},X^{\mathrm{e}}}(\omega) = \zeta^*\omega^{\mathrm{f}}_{\Bbbk} + \zeta^*\omega^{\mathrm{e}}_{\Bbbk} + \zeta^*\omega^{\mathrm{x}}_{\Bbbk}.$$

$$\begin{aligned}
\zeta^*\omega^{\mathrm{f}}_{\Bbbk} &= \Re e\left[(e^{-i\boldsymbol{\theta}\cdot X^{\mathrm{f}}} dz_1 - iz_1 d\boldsymbol{\theta}\cdot X^{\mathrm{f}} e^{-i\boldsymbol{\theta}\cdot X^{\mathrm{f}}}) \wedge (e^{i\boldsymbol{\theta}\cdot X^{\mathrm{f}}} d\bar{z}_2 + i\bar{z}_2 d\boldsymbol{\theta}\cdot X^{\mathrm{f}} e^{i\boldsymbol{\theta}\cdot X^{\mathrm{f}}})\right]\\
&= \Re e\left[dz_1 \wedge d\bar{z}_2 - d\boldsymbol{\theta}\cdot X^{\mathrm{f}} \wedge i(z_1 d\bar{z}_2 + z_2 d\bar{z}_1)\right]\\
&= \omega^{\mathrm{f}}_{\Bbbk} - d\boldsymbol{\theta}\cdot X^{\mathrm{f}} \wedge dq_2.
\end{aligned}$$

$$\begin{aligned}
\zeta^*\omega^{\mathrm{e}}_{\Bbbk} = \sum_{j=1}^{k_{\mathrm{e}}} \Im m \Big[&(e^{-i\boldsymbol{\theta}\cdot(X^{\mathrm{e}}_{j,.})^t} dz^{\mathrm{e}}_j - ie^{-i\boldsymbol{\theta}\cdot(X^{\mathrm{e}}_{j,.})^t} z^{\mathrm{e}}_j d\boldsymbol{\theta}\cdot (X^{\mathrm{e}}_{j,.})^t)\\
&\wedge(e^{i\boldsymbol{\theta}\cdot(X^{\mathrm{e}}_{j,.})^t} d\bar{z}^{\mathrm{e}}_j + ie^{i\boldsymbol{\theta}\cdot(X^{\mathrm{e}}_{j,.})^t} \bar{z}^{\mathrm{e}}_j d\boldsymbol{\theta}\cdot (X^{\mathrm{e}}_{j,.})^t)\Big]\\
= \sum_{j=1}^{k_{\mathrm{e}}} \Im m &\left[dz^{\mathrm{e}}_j \wedge d\bar{z}^{\mathrm{e}}_j - d\boldsymbol{\theta}\cdot (X^{\mathrm{e}}_{j,.})^t \wedge i(z^{\mathrm{e}}_j d\bar{z}^{\mathrm{e}}_j + \bar{z}^{\mathrm{e}}_j dz^{\mathrm{e}}_j)\right]\\
= \omega^{\mathrm{e}}_{\Bbbk} - &\sum_{j=1}^{k_{\mathrm{e}}} d\boldsymbol{\theta}\cdot (X^{\mathrm{e}}_{j,.})^t \wedge dq^{\mathrm{e}}_j.
\end{aligned}$$

$$\begin{aligned}
\zeta^*\omega^{\mathrm{x}}_{\Bbbk} &= d\boldsymbol{\theta} \wedge d(\mathbf{I} + Q^{\mathrm{e}}_{\Bbbk}\cdot (X^{\mathrm{e}})^t + q_2 (X^{\mathrm{f}})^t)\\
(\text{ with formula 3.1})\quad &= \omega^{\mathrm{x}}_{\Bbbk} + d\boldsymbol{\theta}\cdot X^{\mathrm{f}} \wedge dq_2 + \sum_{j=1}^{k_{\mathrm{e}}} d\boldsymbol{\theta}\cdot (X^{\mathrm{e}}_{j,.})^t \wedge dq^{\mathrm{e}}_j.
\end{aligned}$$

So when we sum $\zeta^*\omega^{\mathrm{f}}_{\Bbbk}$, $\zeta^*\omega^{\mathrm{e}}_{\Bbbk}$ and $\zeta^*\omega^{\mathrm{x}}_{\Bbbk}$, we get that $\zeta^*\omega = \omega$.

Now for η, we can again treat separately the action on the different types of E_i's, and just treat the case when the ϵ's are $= -1$. Since η is the identity on the elliptic components, we have $E^*_{1,2}\omega_{\Bbbk} = E^*_{1,2}\omega^{\mathrm{f}}_{\Bbbk} + \omega^{\mathrm{e}}_{\Bbbk} + E^*_{1,2}\omega^{\mathrm{x}}_{\Bbbk}$ and:

$$\begin{aligned}
E^*_1\omega^{\mathrm{f}}_{\Bbbk} &= d\hat{x}_1 \wedge d\hat{\xi}_1 + d\hat{x}_2 \wedge d\hat{\xi}_2 + d\hat{\theta}_3 \wedge d\hat{\xi}_3\\
&= d\xi_1 \wedge d(-x_1) + d\xi_2 \wedge d(-x_2) + d\theta_3 \wedge d\xi_3 = \omega^{\mathrm{f}}_{\Bbbk}\\
E^*_2\omega^{\mathrm{f}}_{\Bbbk} &= d\tilde{x}_1 \wedge d\tilde{\xi}_1 + d\tilde{x}_2 \wedge d\tilde{\xi}_2 + d\tilde{\theta}_3 \wedge d\tilde{\xi}_3\\
&= dx_2 \wedge d\xi_2 + dx_1 \wedge d\xi_1 + d\theta_3 \wedge d\xi_3 = \omega^{\mathrm{f}}_{\Bbbk}.
\end{aligned}$$

Lastly, the transformation $(\boldsymbol{\theta}, \mathbf{I}) \mapsto (\boldsymbol{\theta}\cdot (X^{\mathrm{x}})^{-1}, \mathbf{I}\cdot (X^{\mathrm{x}})^t)$ is a linear symplectomorphism with respect to the symplectic form $\omega^{\mathrm{x}}_{\Bbbk} = \sum_{j=1}^{k_{\mathrm{x}}} d\theta_j \wedge dI_j = d\boldsymbol{\theta} \wedge d\mathbf{I}$. $\square$

The symplectomorphisms ζ and η of the theorem are admissible modifications of semi-toric local models. If we have two local models (φ, G) and (φ', G'), Theorem 3.2.7 tells us we can always modify one of them to get another local model $(\tilde{\varphi}', \tilde{G}')$ such that:

$$\tilde{B} = \tilde{G'}^{-1} \circ G = \begin{pmatrix} id_2 & 0 & 0 \\ 0 & \boldsymbol{\epsilon}^{\mathrm{e}} & 0 \\ 0 & 0 & id_{k_{\mathrm{x}}} \end{pmatrix} + (u, 0, \ldots, 0) \text{ with } u \in \mathcal{F}\ell_S^{\infty}(U).$$

3.2.3 Location of semi-toric critical values

In this subsection, we show how we can extract information about the whole locus of semi-toric critical values from the results we have on the local models. Here, we take the special case of $2n = 6$, and so $F(M)$ is considered as a subset of $\mathbb{R}^3$ taken as an affine space:

Definition 3.2.9. *An embedded curve of focus-focus-transverse critical points in $F(M)$ is called a "nodal path", and it is denoted by γ_i.*

Theorem 3.2.10. *Let F be a semi-toric integrable system on a compact symplectic manifold M^6. Then we have the following statements:*

1. *The locus of focus-focus-transverse critical points is a finite union of nodal paths:*

$$\mathrm{CrP}_{FF-X}(M) = \bigcup_{i=1}^{m_{\mathrm{f}}} \gamma_i,$$

2. *The locus of focus-focus-elliptic critical values is a finite number of points in $F(M)$:*

$$\mathrm{CrP}_{FF-E}(M) = \bigsqcup_{i=1}^{m'_{\mathrm{f}}} c_i,$$

3. *For each γ_i, there exists an affine plane of the form $\mathcal{P}(\gamma_i) = A + \mathbb{R} \cdot e_1 + \mathbb{R} \cdot e_2$ with $e_1 = (1, 0, 0)$ and $e_2 = (0, b, d)$, $b, d \in \mathbb{Z}$ (and $b^2 + d^2 \neq 0$), such that $\gamma_i \subseteq \mathcal{P}(\gamma_i) \cap F(M)$, and in $\mathcal{P}(\gamma_i)$, γ_i is the graph of a smooth function from an interval $]\alpha, \beta[$:*

$$\gamma_i = \{A + t \cdot \vec{e_1} + h(t) \cdot \vec{e_2} \ , \ t \in]\alpha, \beta[\}.$$

   The limits $\gamma(\alpha^+) = \lim_{\substack{t \to \alpha \\ t > \alpha}} \gamma_i(t)$ and $\gamma(\beta^-) \lim_{\substack{t \to \beta \\ t < \beta}} \gamma_i(t)$ are focus-focus-elliptic critical values.

4. *If we assume that the fibers are connected, then the nodal paths are isolated in the sense that there exists a open neighborhood $\mathcal{U}(\gamma_i)$ of γ_i such that the only critical values in $\mathcal{U}(\gamma_i)$ are γ_i.*

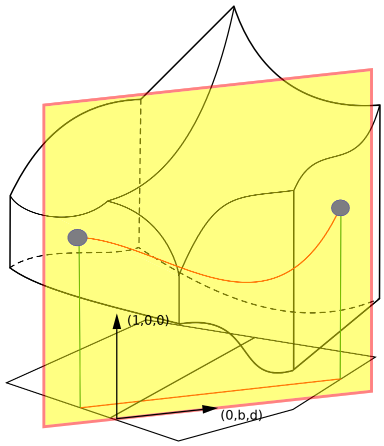

Figure 3.2: A nodal path of FF-X critical values, with its affine plane

In particular, this theorem answers negatively to a question asked to me by Colin de Verdière in 2010: “Can we have a “loop” of focus-focus-transverse singular values in dimension $2n = 6$?”. We must thank him deeply for this simple question that acted both as a compass and as an incentive in my research during the year 2010-2011. We first developped the techniques of local models to answer this question, and then figured out that we could generalize the result using more “conceptual” theorems: this is the object of Chapter 4.

An immediate consequence of Item 3. and 4. is that if the fibers are connected, we have $m'_{\mathrm{f}} = 2m_{\mathrm{f}}$.

In the theorem above, we speak of critical values of a given Williamson type, but we already mentioned in Section 3.1.2 that the Williamson type of a fiber is not well defined. As a result, a critical value can belong to different $\mathrm{CrV}_{\Bbbk}(M)$'s. Yet, we chose to give a result describing the image of the moment map rather than the base space of the foliation, because our “local model” results describe the former while for the latter requires the introduction of new structures and a study on its own. This is what we actually do in Chapter 5. Another reason is that the image of the moment map is the space that physicists directly have access to by experimentation.

Proof of Theorem. 3.2.10

Items 1. and 3.:

Let p be a $FF{-}X$ critical point of a semi-toric integrable system (M, ω, F). Applying Theorem 3.2.3 with the correct system of local coordinates φ in a neighborhood $\mathcal{U}$ of p, we have a smooth function G and a matrix $A \in GL_{n-1}(\mathbb{Z})$ such that $F \circ \varphi = (G_1(Q_{\Bbbk}), A \circ \check{Q}_{\Bbbk})$. The locus of $FF - X$ critical leaves is $\varphi(\{q_1 = q_2 = 0\}) \cap \mathcal{U}$: its image by F is the curve $\gamma_{\mathcal{U}} := F \circ \varphi\left((\{q_1 = q_2 = 0\}) \cap \mathcal{U}\right)$. The curve $\gamma_{\mathcal{U}}$ is parametrized as

$$\{(G_1(0, 0, t), bt, dt) \mid t \in [-\epsilon, \epsilon]\}$$

and it is a regular parametrization:

$$||\gamma'_{\mathcal{U}}(t)|| = \sqrt{\partial_3 G_1(0,0,t)^2 + b^2 + d^2} \geqslant \sqrt{b^2+d^2} > 0 \text{ as } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

The curve $\gamma_{\mathcal{U}}$ is contained in the following affine plan of $\mathbb{R}^3$:

$$\mathcal{P}(\gamma_{\mathcal{U}}) = F(p) + \mathbb{R} \cdot (1,0,0) + \mathbb{R} \cdot (0,b,d).$$

In this plane, if we take $(0,b,d)$ for the x-axis unit vector and $(1,0,0)$ for the y-axis unit vector, $\gamma_{\mathcal{U}}$ is the graph of

$$\begin{aligned} \gamma_{\mathcal{U}} : [0,1] &\to F(M) \\ t &\mapsto G_1(0,0,\frac{t}{\sqrt{b^2+d^2}}) \end{aligned}.$$

This proves item **1.** locally: to a $FF-X$ critical value c we can associate an embedded segment $\gamma_{\mathcal{U}}$ of $FF-X$ critical values such that $c \in \gamma$. Note however, that the association is 1:1 when we consider a $FF-X$ critical point, but not for the $FF-X$ critical value: there can be *a priori* an arbitrary number of $\gamma_{\mathcal{U}}$'s to associate to a c. Yet, we did make the assumption 3.1.10, that we'd only consider simple and topologically stable singularities, so that should rule out this possibility.

There is another argument that goes in favor of our description of the image of the moment map. With Lemma 3.2.2 and Eliasson normal form, we have that the set $\mathrm{CrP}_{FF-X}(F(F(\mathcal{U})))$ is necessarily a finite disjoint union of cylinders. Hence, in $F(\mathcal{U}) \simeq \mathbf{D}^2 \times [-\epsilon,\epsilon]$, there can be only a finite number of nodal paths. In a fiber with more than one leaf with $FF-X$ critical points, there are finitely many $\gamma_{\mathcal{U}}$ that intersect at c but for a given c, we can always pick a $FF-X$ critical point in $F(c)$ and then take the associated $\gamma_{\mathcal{U}}$. That's what we will do for now.

This defines nodal paths only locally near a $FF-X$ critical value. Defining it globally is easy: for a $FF-X$ critical value c, we choose a $FF-X$ critical point p in $F^{-1}(c)$, and then take γ as the image of the connected component of $\mathrm{CrP}_{FF-X}(M)$ that contains p. We have checked that γ verifies the properties of **1.** locally, and we show now that they hold at a global scale.

Let's consider two points c and c' in γ, and their respective local models $(\mathcal{U},\varphi,G)$ and $(\mathcal{U}',\varphi',G')$ as in Theorem 3.2.3. If their intersection is not empty, we have with the relation 3.7 that the planes $\mathcal{P}(\gamma_{\mathcal{U}})$ and $\mathcal{P}(\gamma_{\mathcal{U}'})$ must be the same. We can actually use the surjectivity Theorem 3.2.7 to modify the symplectomorphismone of the local models so that we get an extension of the regular parametrization. Thus, on the extended domain, the graph property comes automatically.

Items 2. and 3.:

For a $FF-E$ point p of a semi-toric integrable system, using Theorem 3.2.3 we have a local model $(\mathcal{U},\varphi,G)$ such that $F \circ \varphi = G(q_1,q_2,q_e)$.

The $FF-E$ critical locus is $\{q_1 = q_2 = q_e = 0\}$: its image by F is a point. Applying Eliasson normal form to it gives an open neighborhood of p with no other $FF-E$ point in it. If we now take for each $FF-E$ points its respective open neighborhood and complete it to get an open covering of M, we can by compactness extract a finite sub-covering of it. As a result, there can be only a finite number of $FF-E$ critical points, and thus only a finite number of $FF-E$ critical values.

In the local model, we can see that $\{q_1 = q_2 = 0\ ,\ q_e \neq 0\}$ is a $FF-X$ critical locus, and $\{q_e = 0\}$ is a $E-X-X$ critical locus. This proves that for a $FF-E$ critical value, there are $FF-X$ and $E-X-X$ critical value locci such that the $FF-E$ critical value is at the intersection of the closure of the $FF-X$ critical locus and the $E-X-X$ critical locus.

Now, given a nodal path γ parametrized by the function of the interval $]0,1[$, its limit $\gamma(0^+)$ in 0^+ or $\gamma(1^-)$ in 1^- exists since $F(M)$ is compact: they are points in the closure of γ. We show this must be the image of a $FF-E$ critical point. First, $\gamma(0^+)$ and $\gamma(1^-)$ must be critical values: a consequence of Liouville-Arnold-Mineur Theorem 1.2.14 is that around a regular value there exists a ball of regular values. Lastly, there is only a finite number of different local models for critical values. One can check that the only one in which there is a critical point in the closure of a $FF-X$ locus is the $FF-E$ local model.

Item 4.:

If we suppose that the fibers are connected, the choice of a nodal path containing a given $FF-X$ critical value is now unique, and we have seen that if we name Γ_p the connected component of $\mathrm{CrP}_{FF-X}(M)$ that contains p, there exists an open neighborhood $\mathcal{U}(\Gamma_p)$ of Γ_p such that $\mathrm{CrP}_{FF-X}(M) \cap \mathcal{U}(\Gamma_p) = \Gamma_p$. Taking the image by F gives the result.

$\square$

In Chapters 4 and 5, we give two different proofs that the fibers of semi-toric moment map are actually connected. Hence, the choice of a nodal path is unique for a given $FF-X$ critical value. This is still a good illustration of the techniques available that may also be used in higher complexity cases to produce results where we don't have for now a result concerning the connectedness of the fibers.

3.3 Action-angle coordinates around (but not on) semi-toric singularities

Remembering that we now work under Assumption 3.1.10, we define here action-angle coordinates on an open set of regular values near a $FF-X^{n-2}$ critical value. We show that if the open set is not simply connected, then one of the coordinates is multivalued. We also give the asymptotics of the coordinates near the critical point.

Theorem 3.3.1. *Let $(M^{2n}, \omega, F = (f_1, \dots, f_n))$ be a semi-toric integrable system with n degrees of freedom. Let m be a $FF - X^{n-2}$ critical point and Λ a regular leaf near m. We assume that $\mathrm{CrL}_{\Bbbk}(M)$ has only one connected component, so $\Gamma = \mathrm{CrV}_{FF-X^{n-2}}(M)$. Let U be an open set of regular values, such that Γ intersects $\bar{U}$.*

There exists a local diffeomorphism

$$G : (V, \{0\} \times \{0\} \times]-\epsilon, \epsilon[^{n-2}) \to (U, \Gamma),$$

a tubular neighborhood $\mathcal{V}$ containing Λ a regular leaf near m and such that $F(\mathcal{V}) = U$, and a symplectomorphism $\varphi : \mathbb{D}^n \times \mathbb{T}^n \to \mathcal{V}$ such that, when we write the coordinates $(\theta_1, \dots, \theta_n, I_1, \dots, I_n)$ on $\mathbb{D}^n \times \mathbb{T}^n$, we have

- *The $(\theta_3, \dots, \theta_n, I_3, \dots, I_n)$ coincide with the transverse action-angle coordinates defined by Theorem 3.1.2 on a neighborhood of m stable by the transverse flows.*
- *We have that*

$$\begin{aligned} I_1(\mathbf{v}) &= S_1(\mathbf{v}) - \mathfrak{Re}(w \ln(w) - w) \\ I_2(\mathbf{v}) &= S_2(\mathbf{v}) + \mathfrak{Im}(w \ln(w) - w) \\ I_3 &= v_3 \\ &\vdots \\ I_n &= v_n \end{aligned}$$

  where $\mathbf{v} = G \circ F$, $w = v_1 + i v_2$, $\ln$ is a determination of the complex logarithm on $U_{v_1,v_2} = U \cap \{v_3, \dots, v_n = 0\} \subseteq \mathbb{R}^2 \simeq \mathbb{C}$ and S_1 and S_2 are smooth functions.

Remark 3.3.2. *The fact that $\Gamma = \mathrm{CrV}_{FF-X^{n-2}}(M)$ is a consequence of Theorem 3.2.10.*

Proof of Theorem. 3.3.1

To prove the theorem we follow and adapt each step of the proof of Section 3 in [VN03] to our case.

Let $F : M^{2n} \to \mathbb{R}^n$ be a semi-toric integrable system, m a $FF - X^{n-2}$ critical point. We already assumed that $\mathrm{CrL}_{FF-X^{n-2}}(M)$ and $\mathrm{CrV}_{FF-X^{n-2}}(M)$ have only one connected component. By Miranda-Zung theorem, there exists $\mathcal{U}_{MZ}$ a neighborhood of m, $\mathcal{U}_{MZ}$ stable by the flow of $\check{F}$, such that, on $\mathcal{U}_{MZ}$ there is a symplectomorphism $\varphi : L_{FF-X^{n-2}} \subseteq T^*\mathbb{R}^2 \times T^*\mathbb{T}^{n-2} \to \mathcal{U}_{MZ} \subseteq M$ and such that, for $Q_{FF-X^{n-2}} = (q_1, q_2, \xi_1, \dots, \xi_{n-2})$,

$$F \circ \varphi = G \circ Q_{FF-X^{n-2}}$$

With Theorem 3.2.10, we denote the nodal path of $FF - X^{n-2}$ values in the image by Γ. Now let's have a point $A_0 \in \mathcal{U}_{MZ} \cap \Lambda_v$ different than m: A_0

is on the same critical fiber as m, and near enough so that Miranda-Zung can be used. We then set Σ^n a (small enough) n-dimensional submanifold which intersects transversally the foliation $\mathcal{F}$ at A_0. We set:

$$\Omega := \{\Lambda_c \in \mathcal{F} \mid \Lambda_c \cap \Sigma^n \neq \emptyset\}.$$

We have that $G \circ F$ is in the same semi-toric equivalence class than F. In particular it is a global moment map for the same foliation $\mathcal{F}$ on the whole "tubular" neighborhood Ω of the leaf containing m (the term "tubular neighborhood" is improper in this case, as the normal bundle is not defined on the pinch of $\check{\mathbb{T}}^2 \times \mathbb{T}^{n-2}$; see [Zun03] for more details). On $\mathcal{U}_{MZ}$, $G \circ F = (q_1, q_2, \xi_1, \ldots, \xi_{n-2})$, so $G \circ F$ is an extention of $Q_{FF-X^{n-2}}$ to Ω. We can now forget G and just take F being a global momentum map that extends $Q_{FF-X^{n-2}}$, and restrict the system to Ω.

A consequence of the fact that the f_1-orbits are homoclinic orbits is that since q_2, ξ_1,...,ξ_{n-2} have 2π-periodic flows on $\mathcal{U}_{MZ}$ and since for their extension $f_2, \ldots, f_n$, all the f_i's Poisson-commute, $\check{F}$ yield a $\mathbb{T}^{n-1}$-action on Ω that commutes with the flow of f_1, and it is free everywhere on Ω except on the cylinder of critical points.

For any point $A \in \Lambda_v$ a regular fiber, we can define $\tau_1(v) > 0$ the time of first return of the χ_{f_1}-flow through the $\mathbb{T}^{n-1}$-orbit of A. If we call this intersection A', there exists a unique couple $(\tau_2(v), \ldots, \tau_n(v)) \in (\mathbb{R}/2\pi\mathbb{Z})^{n-1}$ such that

$$\phi_{f_2}^{\tau_2} \circ \cdots \circ \phi_{f_n}^{\tau_n}(A') = A$$

This is the multi-time needed to close the trajectory of f_1 with the flows of $f_2, f_3, \ldots, f_n$, after one return on the $\mathbb{T}^{n-1}$-orbit of the starting point. Since the joint flow of F is transitive, these times depends only of the Lagrangian torus and thus, only of v and not of A.

For any regular value v, the set of all the $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that $\alpha_1\chi_{f_1} + \cdots + \alpha_n\chi_{f_n}$ has a 1-periodic flow is a sublattice of $\mathbb{R}^3$ called the period lattice. The following matrix

$$\begin{pmatrix} \tau_1(v) & \cdots & \cdots & \cdots & \tau_n(v) \\ 0 & 2\pi & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 2\pi \end{pmatrix}$$

forms a $\mathbb{Z}$-basis of the period lattice (we know that at least $\tau_1 > 0$). These vectors can also be seen as a basis of cycles of the Lagrangian tori Λ_v. The next proposition proves the second item of Theorem 3.3.1: it gives the singular behavior of the basis as $\mathbf{v}$ tends to $\Gamma \cap \bar{U}$.

Proposition 3.3.3. *Let's fix a determination of the complex logarithm : $\ln(w)$, where $w = v_1 + iv_2$. Then the following quantities*

– $\sigma_1(\mathbf{v}) = \tau_1(\mathbf{v}) + \Re e(\ln(w)) \in \mathbb{R}$

– $\sigma_2(\mathbf{v}) = \tau_2(\mathbf{v}) - \Im m(\ln(w)) \in \mathbb{R}/2\pi\mathbb{Z}$

– $\sigma_3(\mathbf{v}) = \tau_3(\mathbf{v}) \in \mathbb{R}/2\pi\mathbb{Z}$, ... , $\sigma_n(\mathbf{v}) = \tau_n(\mathbf{v}) \in \mathbb{R}/2\pi\mathbb{Z}$

can be extended to smooth and single-valued functions of $\mathbf{v}$ in $\bar{U}$. Moreover, the differential form

$$\sigma = \sum_{i=1}^{n} \sigma_i dv_i$$

is closed in $\bar{U}$.

Proof. We fix some $\varepsilon > 0$ and we set

$$\Sigma_u^\alpha = \{z_1 = \varepsilon, z_2 \text{ small }, \boldsymbol{\theta} = (\theta_1, \dots, \theta_{n-2}) = \boldsymbol{\alpha}, \boldsymbol{\xi} \in [-\beta, \beta]^{n-2}\} \subseteq \mathcal{U}_{MZ}$$

$$\Sigma_s^\alpha = \{z_1 \text{ small }, z_2 = \varepsilon, \boldsymbol{\theta} = (\theta_1, \dots, \theta_{n-2}) = \boldsymbol{\alpha}, \boldsymbol{\xi} \in [-\beta, \beta]^{n-2}\} \subseteq \mathcal{U}_{MZ}.$$

These are stable and unstable local submanifolds for the hyperbolic dynamic near the critical point

$$m_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}} = (z_1 = 0, z_2 = 0, \boldsymbol{\theta} = (\theta_1, \dots, \theta_{n-2}) = \boldsymbol{\alpha}, \boldsymbol{\xi}).$$

They are n-dimensional submanifolds intersecting transversally the foliation $\mathcal{F}$ on Ω. Thus, the intersections $A(v, \boldsymbol{\alpha}) := \Lambda_v \cap \Sigma_u^{\boldsymbol{\alpha}}$ and $B(v, \boldsymbol{\alpha}) := \Lambda_v \cap \Sigma_s^{\boldsymbol{\alpha}}$ are points of M in the same χ_{f_1}-orbit. They are well-defined (single-valued) as functions of v, of whom they depend smoothly.

The $\mathbb{T}^{n-1}$-orbits of $A(v, \boldsymbol{\alpha})$ and $B(v, \boldsymbol{\alpha})$ are transversal to the Hamiltonian flow of f_1, thus one can define $\tau_1^{A,B}(v, \boldsymbol{\alpha})$ as the time necessary through the Hamiltonian flow of f_1 starting at $A(v, \boldsymbol{\alpha})$ (which flows outside of $\mathcal{U}_{MZ}$), to make first hit to the $\mathbb{T}^{n-1}$-orbit of $B(v, \boldsymbol{\alpha})$. Let's call this first hit $B' = (b_1', b_2', \boldsymbol{\alpha} + \boldsymbol{\theta}, \boldsymbol{\xi})$. Since the $\mathbb{T}^{n-1}$-orbit of B is in $\mathcal{U}_{MZ}$, we know that in it $f_2 = q_2$, $f_3 = \xi_3$, ...,$f_n = \xi_n$, so we have the explicit expression for the time needed to get back to B, which we call $\tau_2^{A,B}(v, \alpha)$, ... , $\tau_n^{A,B}(v, \alpha)$:

$$\tau_2^{A,B}(v, \boldsymbol{\alpha}) = \arg(b_1') \text{ and for } j = 3, \dots, n \text{ , } \tau_j^{A,B}(v, \boldsymbol{\alpha}) = 2\pi - \theta_j'$$

So here since the f_i's commute, we have that the $\tau_j^{A,B}$ are smooth single-valued functions of $(v, \boldsymbol{\alpha})$, and we actually have that they don't depend of $\boldsymbol{\alpha}$ either. We can now interchange the roles of A and B, and thus, of $\Sigma_u^{\boldsymbol{\alpha}}$ and $\Sigma_s^{\boldsymbol{\alpha}}$, to define the times $\tau_j^{B,A}(v)$ for $j = 1, \dots, n$. The joint flow of F now takes place inside $\mathcal{U}_{MZ}$ where there is a whole codimension-2 manifold of critical leaves $F^{-1}(\Gamma) \cap \bar{\Omega}$. For $\mathbf{v} \in \Gamma$, the quantities $\tau_1^{B,A}(v), \tau_2^{B,A}(v), \dots, \tau_n^{B,A}(v)$ cannot be defined a priori.

However, one should first note that in the definition, $\tau_1^{B,A}(v)$ and $\tau_2^{B,A}(v)$ do not depend actually of the value of $(f_3, \dots, f_n)$: in the Miranda-Zung theorem we use, the local model is a *direct product* of the Eliasson normal form for the focus-focus and the action-angle coordinate for the transversal component. Moreover, since everywhere it is defined, for $j = 3, \dots, n, \tau_j^{B,A}(v) = 0$, its limit when $\mathbf{v} \to \Gamma$ must be 0 also.

With the explicit formulaes of the Hamiltonian flow of q_1 and q_2, we know that $\tau_1^{B,A}(v)$ and $\tau_2^{B,A}(v)$ verify the following equation:

$$(e^{\tau_1^{B,A}+i\tau_2^{B,A}} b_1, e^{-\tau_1^{B,A}+i\tau_2^{B,A}} b_2, \boldsymbol{\theta}, \boldsymbol{\xi}) = (a_1, a_2, 0, \dots, 0, 0, \dots, 0). \tag{3.9}$$

We also have the equations: $a_1 = \varepsilon$, $b_2 = \varepsilon$ and $\bar{a_1} a_2 = \bar{b_1} b_2 = w$. Here we introduce our determination of the complex logarithm to give the solution of 3.9:

$$\tau_1^{B,A} + i\tau_2^{B,A} = \ln(\frac{a_1}{b_1}) = \ln(\varepsilon . \frac{\varepsilon}{\bar{w}}) = \ln(\varepsilon^2) - \ln(\bar{w}).$$

Writing now $\tau_1 + i\tau_2 = (\tau_1^{A,B} + \tau_1^{B,A}) + i(\tau_2^{A,B} + \tau_2^{B,A})$, we can refer to the statement announced on Proposition 3.3.3 concerning σ_1 and σ_2:

$$\begin{aligned}
\sigma_1 + i\sigma_2 &= \tau_1(v) + \Re e(\ln(w)) + i(\tau_2(v) - \Im m(\ln(w))) \\
&= \tau_1^{A,B} + i\tau_2^{A,B} + \tau_1^{B,A} + i\tau_2^{B,A} + \ln(\bar{w}) \\
&= \tau_1^{A,B} + i\tau_2^{A,B} + \ln(\varepsilon^2) - \ln(\bar{w}) + \ln(\bar{w}) \\
&= \tau_1^{A,B} + i\tau_2^{A,B} + \ln(\varepsilon^2).
\end{aligned}$$

This last quantity is smooth with respect to v. Since for $j = 3, \dots, n$, $\sigma_j(\mathbf{v}) = \tau_j(\mathbf{v}) = \tau_j^{A,B}(\mathbf{v})$ is also smooth, this shows the first statement of Proposition 3.3.3.

Let's now show that for regular values, the 1-form $\tau = \sum_{i=1}^n \tau_i dv_i$ is closed. For this we fix a regular value $\mathbf{v} \in U$ and introduce the following action integral

$$\mathcal{A}(\mathbf{v}) = \int_{\gamma_v} \alpha$$

where α is any Liouville 1-form of ω on a tubular neighborhood of $\mathcal{V}(\Lambda_{\mathbf{v}})$ ($\omega = d\alpha$), and $\mathbf{v} \mapsto \gamma_{\mathbf{v}} \subseteq \Lambda_{\mathbf{v}}$ is a smooth family of loops with the same homotopy class in $\Lambda_{\mathbf{v}}$ as the joint flow of F at the times $(\tau_1(\mathbf{v}), \tau_2(\mathbf{v}), \dots, \tau_n(\mathbf{v}))$. The integral $\mathcal{A}$ only depends of $\mathbf{v}$ as $\gamma_{\mathbf{v}} \subseteq \Lambda_{\mathbf{v}}$, which is Lagrangian (this is another statement of Mineur's formula).

A consequence of Darboux-Weinstein Theorem 1.3.1 is that we can identify each Lagrangian leaf of $\mathcal{F}$ in $\mathcal{V}(\Lambda_{\mathbf{v}})$ with a closed 1-form on $\Lambda_{\mathbf{v}}$ (see the article [Wei71]). The normal bundle

$$N\Lambda_{\mathbf{v}} = \bigsqcup_{p\in\Lambda_{\mathbf{v}}} \frac{T_pM}{T_p\Lambda_{\mathbf{v}}}$$

can be identified with $T^*\Lambda_{\mathbf{v}}$ using the symplectic form: for $m \in \Lambda_{\mathbf{v}}$ and $X_m \in T_mM$, we define

$$\tilde{\omega}[X_m]_m := (\imath_{X_m}\omega_m)|_{T\Lambda_{\mathbf{v}}} \in T^*_m\Lambda_{\mathbf{v}}.$$

Since $T_mM = T_m\Lambda_{\mathbf{v}} \oplus N_m\Lambda_{\mathbf{v}}$ with $\Lambda_{\mathbf{v}}$ Lagrangian, the map

$$\begin{aligned} TM &\to T^*\Lambda_{\mathbf{v}} \\ X &\mapsto \tilde{\omega}[X] \end{aligned}$$

is linear, and $\tilde{\omega}[X]$ is non-zero if and only if $p_{N\Lambda_{\mathbf{v}}}(X) \neq 0$ as a vector field. Thus, an infinitesimal deformation of $\Lambda_{\mathbf{v}}$ is a vector field of $\mathcal{V}(\Lambda_{\mathbf{v}})$ transversal to $\Lambda_{\mathbf{v}}$, that is, a section of $N\Lambda_{\mathbf{v}}$. Such an infinitesimal deformation is performed in the space of Lagrangian manifolds if and only if the associated 1-form is closed, that is, if the deformation vector field is locally Hamiltonian.

Our foliation here $(\Lambda_{\mathbf{v}})_{\mathbf{v}\in U}$ is given by the fibers of the moment map $F|_\Omega$, thus the deformation 1-form given by the variation of the value v_i of f_i verifies

$$\left.\frac{\partial}{\partial v_i}\Lambda_{\mathbf{v}}\right|_{\mathbf{v}=\mathbf{c}} = \kappa_i(c)$$

where κ_i is the closed 1-form on $\mathcal{V}(\Lambda_{\mathbf{v}})$ defined by: $\imath_{X_{f_j}}\kappa_i = \delta_{i,j}$. In other words, the integral of κ_i along a trajectory of the flow of f_j measures the increasing of time t_j along this trajectory. We now show the following formula linking the variation 1-form to the infinitesimal variation of the action $\mathcal{A}$

$$\frac{\partial\mathcal{A}}{\partial v_i}(c) = \int_{\Gamma_c}\kappa_i \tag{3.10}$$

Thus, the only thing left to do to conclude the proof of Theorem 3.3.1 is to prove that

$$\left.\frac{\partial}{\partial v_i}(\gamma_{\mathbf{v}}^*\alpha)\right|_{\mathbf{v}=\mathbf{c}} \text{ and } \gamma_{\mathbf{c}}^*\kappa_i$$

are cohomologous on S^1. We have

$$\frac{d}{dv_i}(\gamma_{\mathbf{v}}^*\alpha) = \gamma_{\mathbf{v}}^*\left[\mathcal{L}_{\frac{\partial\gamma_{\mathbf{v}}}{\partial v_i}}\alpha\right] = \gamma_{\mathbf{v}}^*\left[\imath_{\frac{\partial\gamma_{\mathbf{v}}}{\partial v_i}}d\alpha + d\imath_{\frac{\partial\gamma_{\mathbf{v}}}{\partial v_i}}\alpha\right].$$

For $\mathbf{c} \in U$, $\Lambda_{\mathbf{c}}$ is Lagrangian and $\frac{\partial\gamma_{\mathbf{v}}}{\partial v_i}|_{\mathbf{v}=\mathbf{c}}$ splits into two components $X^t_{\mathbf{c}}$ and $X^n_{\mathbf{c}}$, with $X^t_{\mathbf{c}} \in T\Lambda_{\mathbf{c}}$ and $X^n_{\mathbf{c}} \in N\Lambda_{\mathbf{c}}$. The normal vector $X^n_{\mathbf{c}}$ is by definition the infinitesimal deformation of $\mathcal{F}$ at $\mathbf{c}$ along the direction $\frac{\partial}{\partial v_i}$, that is, κ_i. We then have

$$\gamma_{\mathbf{c}}^* \left(\underbrace{\imath_{X_{\mathbf{c}}^t} d\alpha}_{=0} + \imath_{X_{\mathbf{c}}^n} d\alpha \right) = \gamma_{\mathbf{c}}^* \tilde{\omega}[\frac{\partial}{\partial v_i}] = \gamma_{\mathbf{c}}^* \kappa_i.$$

Thus $\frac{\partial}{\partial v_i}(\gamma_{\mathbf{v}}^* \alpha)\Big|_{\mathbf{v}=c} - \gamma_{\mathbf{c}}^* \kappa_i = d\alpha(\frac{\partial \gamma_{\mathbf{v}}}{\partial v_i})$ is exact: the two 1-forms are cohomologous. $\square$

Since $\gamma_{\mathbf{v}}$ has the same homotopy class as the joint flow of F at the multi-time $\tau(\mathbf{v})$, we have

$$d\mathcal{A} = \sum_{i=1}^{n} \frac{\partial \mathcal{A}}{\partial v_i} dv_i = \sum_{i=1}^{n} \tau_i dv_i = \tau.$$

Thus, τ is a closed 1-form, $\mathcal{A}$ is the action integral, defined for all $\mathbf{v} \in U$. $\ln(w)$ is also closed as a holomorphic 1-form in w and thus, σ is closed for all regular values of $\mathbf{v}$, and hence, for all $\mathbf{v} \in \bar{U}$, since we can prove that Γ is contractible.

$\square$

Definition 3.3.4. *Let S be the unique smooth function of $\mathbf{v}$ defined on $\bar{U}$ such that $dS = \sigma$. The Taylor serie of S in (v_1, v_2) can be written as*

$$S = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{\partial S}{\partial v_1^{j_1} \partial v_2^{j_2}}(\mathbf{v}).$$

In accordance with [VN03] we call this double sum the symplectic invariant of the nodal locus Γ and we have:

$$\mathcal{A}(\mathbf{v}) = S(\mathbf{v}) - \Re e(w \ln w - w).$$

Now, in the proof of Proposition 3.3.3, we have that even if the “interior” times $\tau_j^{B,A}$ cannot be defined for $\mathbf{v} \in \gamma$, the “exterior” times $\tau_j^{A,B}$ are still defined completely. Since in $\mathcal{U}_{MZ}$ the flow is radial, we know that even if we take the limit of the flow of f_1 when $t \to \pm\infty$, the flow won't swirl. This is what allowed us to prove that $\tau_3, \ldots, \tau_n$ could be smoothly extended to critical values in Γ. We can now look at the map

$$\begin{aligned} &(\mathrm{CrP}_{FF-X}(\bar{\Omega}) \to \mathrm{CrP}_{FF-X}(\bar{\Omega}) \\ &(\theta_3, \ldots, \theta_n, \mathbf{v}) \mapsto (\theta_3 + \tau_3(v), \ldots, \theta_n + \tau_n(v), \mathbf{v}). \end{aligned}$$

It is just the Hamiltonian flow of $\check{F}^{>2}$ at the joint time $\tau^{>2}(\mathbf{v})$, and it is a symplectomorphism of $(\mathrm{CrP}_{FF-X^{n-2}}(\bar{\Omega}), \omega_{FF-X^{n-2}})$ where $\omega_{FF-X^{n-2}}$ is the symplectic form on $\mathrm{CrP}_{FF-X^{n-2}}$ introduced in Theorem 3.1.14. With the integrable system $\check{F}^{>2}|_{\mathrm{CrP}_{FF-X^{n-2}}(\bar{\Omega})}$, we can define on $\gamma = F(\mathrm{CrP}_{FF-X}(\bar{\Omega})$ a period lattice. This is an illustration of the stratified integral affine structure of almost-toric systems (see Chapter 5).

Another way to look at this is to take the limit when $\varepsilon \to 0$ in the definition of $\Sigma_s^{\boldsymbol{\alpha}}$ and $\Sigma_u^{\boldsymbol{\alpha}}$. The map links points of the torus $\mathbb{T}_{\mathbf{v}}^{n-2}$ of critical values of the leaf $\Lambda_{\mathbf{v}}$: we have a set of couples $(p_-, p_+) \in (\mathbb{T}_{\mathbf{v}}^{n-2} \times \mathbb{T}_{\mathbf{v}}^{n-2})_{\mathbf{v}\in\gamma}$ and $\phi_{\check{F}\leqslant 2}^{(\tau_3(\mathbf{v}),\ldots,\tau_n(\mathbf{v}))}(p_-) = p_+$.

The smooth extension of σ_3 gives us immediately the monodromy near a nodal path

Corollary 3.3.5. *With the same hypothesis as Theorem 3.3.1, Γ is of codimension 2, so we can assume the existence of a loop δ going around γ, and oriented positively.*

In the homological basis of Liouville torus given by the n fundamental circles, the topological monodromy matrix is:

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

To summarize, in this chapter, we have provided several results using techniques we believe to be robust enough to be of good help in less friendly settings (in almost-toric systems of higher complexity, for instance). However, the downside will be the increasing "heaviness" of the techniques, where the profusion of notations tends to hide the phenomenons described. In a sense, Chapter 4 and 5 are attempts to keep on moving to always more conceptual frameworks.

An interesting and useful developpement would be to rewrite completely all the definitions and the results obtained so far in the setting of symplectic stradispaces: it would for sure provide us with simple proofs for a lot of results that the community holds to be true but for which the techniques at our disposal are somewhat still too heavy.

Chapter 4

About the image of semi-toric moment map

4.1 Description of the image of the moment map

In the article [VN07], Theorem **3.4** gives a nice description of the image of the moment map of a semi-toric system, along with the connectedness of the fibers of F. We extend here the description to the case of arbitrary dimension, relying on our stratification theorem 3.1.14.

First, we need to prove this lemma we announced in Section 1.2.1.

Lemma 4.1.1. *Let's (M, ω, F) be an integrable Hamiltonian system. It factors through the base space $\mathcal{B}$ by the following commuting diagram:*

$$\begin{tikzcd} M \arrow[r, "\pi_{\mathcal{F}}"'] \arrow[dr, "F"'] & \mathcal{B} \arrow[d, "\tilde{F}"] \\ & F(M) \subset \mathbb{R}^n \end{tikzcd}$$

If there are no hyperbolic critical points, the map $\tilde{F}$ is a local diffeomorphism.

Proof of Lemma. 4.1.1 To define a diffeomorphism, one needs a differential structure on the origin and the target spaces. Here, this step is not trivial as neither $\mathcal{B}$ nor $F(M)$ are manifolds. They are only stratified by manifolds (in particular, they are manifolds with corners). For $F(M)$, it is not problematic because $F(M)$ is a closed subset of $\mathbb{R}^n$, so we can define the set

$$\mathcal{C}^\infty(F(M) \to \mathbb{R}) := \{v \in \mathcal{C}^0(F(M)) \mid \exists \tilde{v} \text{ with } \tilde{v}|_{F(M)} = v\}.$$

For $\mathcal{B}$ however, as it is an intrinsic object, we need to define a differential structure on it *ab initio*.

We have seen with Theorem 3.1.14 that the Williamson type stratifies M by symplectic (sub-)manifolds and this will give us the differential structure of $\mathcal{B}$. The differential structure $\mathbf{C}$ of $\mathcal{B}$ is a subsheaf of the sheaf of continuous function on $\mathcal{B}$. In this case, for an open set V of $\mathcal{B}$, $\mathbf{C}(V)$ is defined as continuous functions that generate system-preserving $\mathbb{R}^1$-actions on $\pi_{\mathcal{F}}^{-1}(V)$: they are the functions $u \in \mathcal{C}^0(V)$ such that $u \circ \pi_{\mathcal{F}} \in \mathcal{C}^\infty(\pi_{\mathcal{F}}^{-1}(V) \to \mathbb{R})$. All these definitions are consistent with the definitions given in [Zun03].

Now, given an open neighborhood U of $F(M)$ and a point $m \in F^{-1}(U)$, we know with Theorem 3.1.11 that there exists a small enough tubular neighborhood $\mathcal{U}_m$ which intersects only one connected component of $F^{-1}(U)$. Hence we can define on U a map $\tilde{F}^{-1}$ which associates to $c \in U$ the point $b \in \mathcal{B}$ associated to the unique leaf $\Lambda_b = F^{-1}(U) \cap \mathcal{U}_m$. This will define the inverse map of $\tilde{F}$ on the small open set $\pi_{\mathcal{F}}(\mathcal{U}_m)$.

We still need to show that both $\tilde{F}$ and its inverse are smooth. For $\tilde{F}$, that's easy : $\tilde{F} \circ \pi_{\mathcal{F}} = F$ by definition and it is a smooth function from M to $\mathbb{R}^n$, so $\tilde{F} \in \mathbf{C}$. For $\tilde{F}^{-1}$ it is a bit harder: we need to show that for any $u \in \mathbf{C}$, $u \circ \tilde{F}^{-1} \in \mathcal{C}^\infty(U \to \mathbb{R})$. Now, since $u \in \mathbf{C}$ we have $u \circ \pi_{\mathcal{F}} =: f \in \mathcal{C}^\infty(\mathcal{U}_m \to \mathbb{R})$ and $\{f, f_i\} = 0$ for $i = 1, \dots, n$. Because there are no hyperbolic singularities in our system, with Eliasson normal form and Lemma 3.2.1, we know that $f = g \circ F$ with $g \in \mathcal{C}^\infty(\mathbb{R}^n \to \mathbb{R})$, and thus, $u \circ \tilde{F}^{-1}$ is simply equal to g, which is smooth as desired.

□

4.1.1 Atiyah – Guillemin & Sternberg revisited

We propose a refined version (or an extension, depending on the point of view) of Atiyah – Guillemin & Sternberg theorem in the case of an almost toric system. We take the usual affine structure of $\mathbb{R}^n$. Here π_Δ will denote the projection on the last $n-1$ coordinates, from $F(M)$ to $\check{F}(M)$. In this section we will describe the locci of critical values using the classical Atiyah – Guillemin & Sternberg theorem and its consequences. We first recall two theorems.

Theorem 4.1.2. *Let (M^{2n}, ω, F) be a semi-toric system. We do not require the $\mathbb{T}^{n-1}$-action to be effective. We have the following results:*

1. **Atiyah – Guillemin & Sternberg:** *$\check{F}(M) =: \Delta$ is a rational convex polytope. The fibers of $\check{F}$ are connected.*
2. **Theorem 3.1.14:** *Locci of critical* points *of a given Williamson type are unions of connected manifolds of dimension k_{x}:*

$$\mathrm{CrP}_{\Bbbk}(M) = \bigcup_i A_{\Bbbk}^{(i)}.$$

Each $A_{\Bbbk}^{(i)}$ can be endowed with a natural symplectic form $\omega_{\Bbbk}^{(i)}$.

Remark 4.1.3. *When we will prove the connectedness of the fibers with Theorem 4.2.4, we will prove that in the last assertion of the theorem the union is disjoint.*

Here we have considered the strata, but another natural collection of sets to consider are the skeleta

$$\tilde{A}_{\Bbbk}^{(i)} = \bigcup_{A_{\Bbbk'}^{(j)} \leq A_{\Bbbk}^{(i)}} A_{\Bbbk'}^{(j)}$$

of the connected components $A_{\Bbbk}^{(i)}$ of $\mathrm{CrP}_{\Bbbk}(M)$, the set of critical points of a given $\Bbbk$ on the manifold M. We have the following theorem:

Theorem 4.1.4. Semi-toric skeleta in dimension 6
Let (M^6, ω, F) be a semi-toric integrable system of dimension 6.

The skeleta $\tilde{A}_{\Bbbk}^{(i)}$ of the connected components $A_{\Bbbk}^{(i)}$ of $\mathrm{CrP}_{\Bbbk}(M)$ are smooth manifolds. They can be endowed with a natural symplectic form $\tilde{\omega}_{\Bbbk}^{(i)}$.

Let $F_{\Bbbk}^{(i)} := F \circ i_{\tilde{A}_{\Bbbk}^{(i)}}$. Then

$$\forall i \in \mathcal{W}_0(F),\ (\tilde{A}_{\Bbbk}^{(i)}, \tilde{\omega}_{\Bbbk}^{(i)}, F_{\Bbbk}^{(i)}) \textit{ is a semi-toric system of dimension } 2k_{\mathrm{x}}.$$

While it was rather easy to show that the strata were manifolds, it is much harder to show that result for the skeleta. The consequence is that we fully prove Theorem 4.1.4, the symplectic manifold and the semi-toric character of the set, in dimension $2n = 6$ while we can only conjecture it in higher dimension for now.

Nevertheless, we still prove that for a semi-toric integrable system of any dimension, all skeleta are smooth submanifolds. We also have good hope to prove that they are in fact semi-toric integrable systems. Hence, we make the following conjecture:

Conjecture 4.1.5. Semi-toric skeleta in dimension 2n
Let (M, ω, F) be a semi-toric integrable system of dimension $2n$.

The skeleta $\tilde{A}_{\Bbbk}^{(i)}$ of the connected components $A_{\Bbbk}^{(i)}$ of $\mathrm{CrP}_{\Bbbk}(M)$ are smooth manifolds. They can be endowed with a natural symplectic form $\tilde{\omega}_{\Bbbk}^{(i)}$.

Let $F_{\Bbbk}^{(i)} := F \circ i_{\tilde{A}_{\Bbbk}^{(i)}}$. Then

$$\forall i \in \mathcal{W}_0(F),\ (\tilde{A}_{\Bbbk}^{(i)}, \tilde{\omega}_{\Bbbk}^{(i)}, F_{\Bbbk}^{(i)}) \textit{ is a semi-toric system of dimension } 2k_{\mathrm{x}}.$$

If Conjecture 4.1.5 is proved to be true, we get with Theorem 4.1.7 that strata given by the connected components of $\mathrm{CrV}_{\Bbbk}^{F}(M)$ are, *most of the time*, graphs of smooth functions: **Item a.** of Theorem 4.1.7 describes the strata. We take the precaution to say that it is true only *most of the time*, because the strata can be vertical with respect to π_Δ. In that case, **Item b.** of Theorem 4.1.7 describes the strata.

In the meantime we give this Proposition which is easy to prove. It takes profit of the $\mathbb{T}^{n-1}$-action that survived in the integrable system, and of the integral affine structure that comes along.

Proposition 4.1.6. *Let's set $B_{\Bbbk}^{(i)} := F(A_{\Bbbk}^{(i)})$ and $\tilde{B}_{\Bbbk}^{(i)} := F(\tilde{A}_{\Bbbk}^{(i)})$. We have that $\tilde{B}_{\Bbbk}^{(i)} = \overline{B_{\Bbbk}^{(i)}}$ and $B_{\Bbbk}^{(i)} \subset \mathbb{R}_{f_1} \times \Delta_{\Bbbk}^{(i)}$, an open affine solid cylinder, vertical with respect to π_Δ and whose basis $\Delta_{\Bbbk}^{(i)} := \check{F}_{\Bbbk}^{(i)}(A_{\Bbbk}^{(i)})$ is an open subpolytope of Δ of dimension $\check{k}_{\mathrm{x}} = rk(\pi_\Delta \circ F \circ i_{\tilde{A}_{\Bbbk}^{(i)}})$.*

Proof of Proposition. 4.1.6 The $A_{\Bbbk}^{(i)}$ are stable submanifolds of M for F and the orbit of $p \in A_{\Bbbk}^{(i)}$ by $\check{F}_{\Bbbk}^{(i)}$ and $\check{F}$ are the same. We have that $\check{F}_{\Bbbk}^{(i)}$ is $\check{k}_{\mathrm{x}}^{(i)}$-toric, so any fiber of $\check{F}_{\Bbbk}^{(i)}$ is connected and made of a single orbit. Now, since F is continuous (it is even smooth), from

$$\left(\pi_\Delta \circ (F|_{A_{\Bbbk}^{(i)}})\right)^{-1}(\check{q}) = \left(\check{F}_{\Bbbk}^{(i)}\right)^{-1}(\check{q})$$

we deduce that

$$F \circ \left((F_{\Bbbk}^{(i)})^{-1}(\check{q})\right) = B_{\Bbbk}^{(i)} \cap (\pi_\Delta)^{-1}(\check{q})$$

is contained in a segment and connected: it is a closed segment in $F(M)$, vertical with respect to f_1, possibly reduced to a point.

Now, applying Atiyah – Guillemin & Sternberg to:

$$\left(A_{\Bbbk}^{(i)}, \omega_{\Bbbk}^{(i)}, \check{F}_{\Bbbk}^{(i)}\right)$$

we have that the fibers of $\check{F}_{\Bbbk}^{(i)}$ *in* $A_{\Bbbk}^{(i)}$ are connected and that $\Delta_{\Bbbk}^{(i)} = \check{F}_{\Bbbk}^{(i)}(A_{\Bbbk}^{(i)})$ is a rational convex polytope. It is a subpolytope of Δ of dimension $\check{k}_{\mathrm{x}}^{(i)}$, because the rank of $\mathbf{T}_{\Bbbk}^{(i)} \circlearrowleft A_{\Bbbk}^{(i)}$ is less than $n-1$, or equivalently, because $\check{F}_{\Bbbk}^{(i)}$ is a sub-system of $\check{F}$. It is open because we have constructed $\check{F}_{\Bbbk}^{(i)}$ as a $\check{k}_{\mathrm{x}}^{(i)}$-toric system without critical points in $A_{\Bbbk}^{(i)}$. Now, since $\pi_\Delta(B_{\Bbbk}^{(i)}) = \Delta_{\Bbbk}^{(i)}$, we get the result that was announced. □

This Proposition simply states that the projection of a strata is a subpolytope of Δ. The next theorem tells us more about the strata themselves, by giving a *complete* description of them. It is the counterpart in higher dimension to Proposition **2.9.** and Theorem **3.4** of [VN07]. In this sense, this result is close to Theorem 3.2.10. We hope all these results to play a key role in the description of semi-toric systems using moment polytopes.

Theorem 4.1.7. Description: *Given the two functions*

$$m_{\Bbbk}^{(i)} : \Delta_{\Bbbk}^{(i)} \to \mathbb{R}_{f_1}, \quad \check{q} \mapsto \min_{(\check{F}_{\Bbbk}^{(i)})^{-1}(\check{q})} f_1 \qquad \text{and} \quad M_{\Bbbk}^{(i)} : \Delta_{\Bbbk}^{(i)} \to \mathbb{R}_{f_1}, \quad \check{q} \mapsto \max_{(\check{F}_{\Bbbk}^{(i)})^{-1}(\check{q})} f_1$$

we have the following alternative:

a. *If $k_{\mathrm{x}}^{(i)} = \check{k}_{\mathrm{x}}^{(i)}$, that is, if $\pi_{\Delta}^{-1}(q)$ intersects transversally $B_{\Bbbk}^{(i)}$ then :*

$$m_{\Bbbk}^{(i)} = M_{\Bbbk}^{(i)}.$$

They are smooth functions, and $B_{\Bbbk}^{(i)}$ is the graph of the first function of the integrable system $f_{\Bbbk,1}^{(i)}$:

$$B_{\Bbbk}^{(i)} = (f_{\Bbbk,1}^{(i)}(\Delta_{\Bbbk}^{(i)}), \Delta_{\Bbbk}^{(i)}) = (m_{\Bbbk,1}^{(i)}(\Delta_{\Bbbk}^{(i)}), \Delta_{\Bbbk}^{(i)}) = (M_{\Bbbk,1}^{(i)}(\Delta_{\Bbbk}^{(i)}), \Delta_{\Bbbk}^{(i)}).$$

b. *If $k_{\mathrm{x}}^{(i)} \neq \check{k}_{\mathrm{x}}^{(i)}$, then $\tilde{B}_{\Bbbk}^{(i)} = epi(m_{\Bbbk}^{(i)}) \cap hypo(M_{\Bbbk}^{(i)})$, where $m_{\Bbbk}^{(i)}, M_{\Bbbk}^{(i)}$ are continuous functions, and $\pi_{\Bbbk}^{(i)} : \tilde{B}_{\Bbbk}^{(i)} \to \Delta_{\Bbbk}^{(i)}$ is a Serre fibration.*

Remark 4.1.8. *Remember the regular values have only one connected component. They are also the (continuous) prolongations of $m_{\Bbbk}^{(i)}$ and $M_{\Bbbk}^{(i)}$.*

Even if we can't prove it for now, we believe the case described in **Item b.** to be rather exceptional. One "constant" exception is $\mathrm{CrV}_{X^n}(M)$, which is always vertical with respect to π_{Δ}: $\mathrm{CrV}_{X^n}(M)$ has always one more dimension than the polytope Δ.

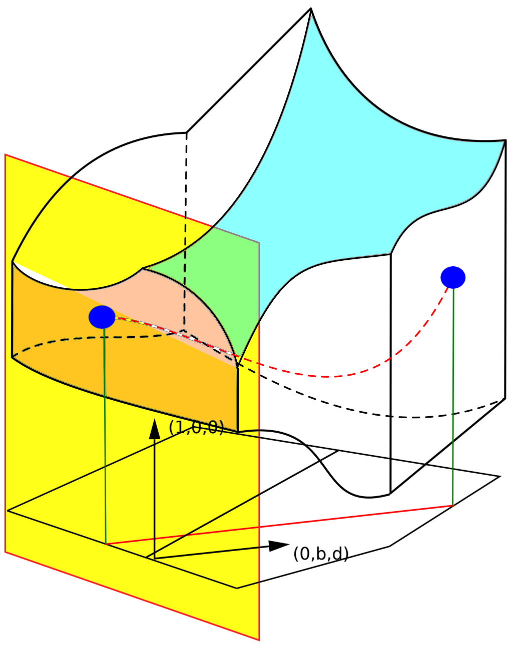

Figure 4.1: Bifurcation diagram, with the two possible cases **a.** and **b.**

Proof of Theorem. 4.1.4 *for* $2n = 6$

We need to prove that the skeleta are also smooth manifolds. First, we give a proof in dimension $2n = 6$ that skeleta are symplectic manifolds of dimension $2k_{\mathrm{x}}$. Then, we give a proof that in arbitrary dimension, the skeleta are smooth symplectic manifolds of dimension $2k_{\mathrm{x}}$. The last thing we prove is that the moment map, restricted to a skeleton of dimension $2n = 4$, is still a semi-toric moment map. We only prove it for skeleta of semi-toric integrable systems of dimension $2n = 6$, but we make the conjecture that it is true in all dimensions.

First the case of the skeleton of the connected component of $\mathrm{CrP}_{X^n}(M)$. First, $\mathrm{CrP}_{X^n}(M)$ is path connected for all dimension, as the critical points are of codimension 2. Second, we are dealing with an integrable Hamiltonian system, so $\mathrm{CrP}_{X^n}(M)$ is of full Liouville measure, and in particular open and dense. Thus, in this case its skeleton is just M, and it is of course a smooth manifold.

Now, back to the case $2n = 6$, there are only a finite number of skeleta to examine:

– 2 cases of dimension 0: (the c.c. of) $\mathrm{CrP}_{E^3}(M)$ and $\mathrm{CrP}_{FF-E}(M)$
– 2 cases of dimension 2: (the c.c. of) $\mathrm{CrP}_{E^2-X}(M)$ and $\mathrm{CrP}_{FF-X}(M)$
– 1 cases of dimension 4: (the c.c. of) $\mathrm{CrP}_{E-X^2}(M)$
– 1 cases of dimension 6: (the c.c. of) $\mathrm{CrP}_{X^3}(M)$

We already treated the 6-dimensional case. The cases of dimension 0 are trivial : their skeleton is themselves.

For a $\mathrm{CrP}_{E^2-X}(M)$, the skeleton is gluing the two $A^{(i)}_{E^3}(M)$ at its ends. We just write the procedure for one. We have to write, in the chart given by Eliasson in the E^3-case, the local coordinates of one of the three $A^{(i)}_{E^2-X}(M)$ that come along a E^3 critical point. Locally, a $A^{(i)}_{E^2-X}(M)$ is diffeomorphic to an open cylinder, and in local coordinates $(x_1^e, y_1^e, x_2^e, y_2^e, x_3^e, y_3^e) \in \mathbb{R}^6$ near $\mathrm{CrP}_{E^3}(M)$, after having left-composed by the diffeomorphism $G^{-1}_{E^3}$ in Theorem 1.2.23, if we set $q_i^e = x_i^2 + y_i^2$, we have for instance that

$$\mathrm{CrP}_{E^2-X}(M) = \{(x_1^e, y_1^e, 0, 0, 0, 0) \mid q_1^e > 0\} \text{ and } \mathrm{CrP}_{E^3}(M) = (0, 0, 0, 0, 0, 0).$$

This gives us a smooth chart of the skeleton from a neighborhood of one of the $\mathrm{CrP}_{E^3}(M)$ to a neighborhood of the origin in $\mathbb{R}^2$. Globally this amounts to glue smoothly two points at the ends of the symplectic cylinder $\mathrm{CrP}_{E^2-X}(M)$, and we get a symplectic sphere.

For a $\mathrm{CrP}_{FF-X}(M)$, taking the skeleton is gluing the two $\mathrm{CrP}_{FF-E}(M)$ at its ends. Again, we just write the procedure for one. Again, we have to write in the $FF - E$ Eliasson chart, the local coordinates of the $\mathrm{CrP}_{FF-X}(M)$

that comes along a $\mathrm{CrP}_{FF-E}(M)$. Locally, a $\mathrm{CrP}_{FF-X}(M)$ is once more diffeomorphic to an open cylinder, and again, after having left-composed by the diffeomorphism G^{-1}, we have in local coordinates $(z_1, z_2, q_{\mathrm{e}}) \in \mathbb{C}^3$ near $\mathrm{CrP}_{FF-E}(M)$ that

$\mathrm{CrP}_{FF-X}(M) = \{(0,0,0,0,x_1^e,y_1^e) \mid q_{\mathrm{e}} > 0\}$ and $\mathrm{CrP}_{E-X-X}(M) = (z_1, z_2, 0, 0) \mid \bar{z}_1 z_2 \neq 0$

So this gives us a smooth chart of the skeleton from a neighborhood of one of the $\mathrm{CrP}_{FF-E}(M)$ to a neighborhood of the origin in $\mathbb{R}^2$. Globally this amounts again to glue smoothly two points at the ends of the symplectic cylinder $\mathrm{CrP}_{FF-X}(M)$ to get a symplectic sphere.

We are now left with the hardest case, the $E-X^2$ case. It is harder because here when we take the skeleton, the submanifolds we glue are no longer trivial: there can be critical points of $FF-E$,E^2-X and E^3 type. For each of them we must examine in Eliasson local coordinates if the gluing is smooth.

First we glue the E^2-X critical points to the $E-X^2$ critical set. In Eliasson local coordinates $(x_1^e, y_1^e, x_2^e, y_2^e, \theta_3, \xi_3)$ near a $\mathrm{CrP}_{E^2-X}(M)$, this amounts locally to pick $A^{(1)}_{E-X^2}$, one of the two 4-dimensional planes that are the connected components of $\mathrm{CrP}_{E-X^2}(M)$ that come along with a E^2-X critical point. After having left-composed by the diffeomorphism $G^{-1}_{E^2-X}$ of Theorem 1.2.23, we have in local coordinates that for instance

$$A^{(1)}_{E-X^2} = \{(0,0,x_2^e,y_2^e,\theta_3,\xi_3) \mid q_2^e > 0\ ,\ \theta_3 \in S^1,\ \xi_3 \in]-\varepsilon,\varepsilon[\}$$

that is, a punctured plane times an open cylinder. We have also

$$A^{(1)}_{E^2-X}(M) = \{(0,0,0,0,\theta_3,\xi_3) \mid \theta_3 \in S^1,\ \xi_3 \in]-\varepsilon,\varepsilon[\}$$

which is a cylinder, as we already saw. They are glued together smoothly to give a plane times an open cylinder

$$\Big\{(0,0,x_2^e,y_2^e,\theta_3,\xi_3) \mid q_2^e \geqslant 0\ ,\ \theta_3 \in S^1\ ,\ \xi_3 \in]-\varepsilon,\varepsilon[\Big\}.$$

Next we look at the E^3 critical points. In Eliasson local coordinates near a $\mathrm{CrP}_{E^3}(M)$, there are three $\mathrm{CrP}_{E-X^2}(M)$. We choose one by picking a $A^{(1)}_{E-X^2}$, one of the three 4-dimensional planes that are the connected components of $\mathrm{CrP}_{E-X^2}(M)$ that come along with a E^3 critical point. After having left-composed by the diffeomorphism G_{E^3} of Theorem 1.2.23, we have in local coordinates that

$$\mathrm{CrP}_{E-X^2}(M) = A^{(1)}_{E-X^2} = \{(0,0,x_2^e,y_2^e,x_3^e,y_3^e) \mid q_2^e > 0\ ,\ q_3^e > 0\}.$$

When we take the skeleton here, it means we glue the set above with the two adjacent E^2-X and the E^3 critical point. We already determined the equation of these sets in local coordinates, and thus the skeleton is

$$A^{(1)}_{E-X^2} \cup A^{(1)}_{E^3} \bigcup_{i=1}^{2} A^{(i)}_{E^2-X} = \{(0,0,x_2^e,y_2^e,x_3^e,y_3^e) \mid q_2^e \geqslant 0\ ,\ q_3^e \geqslant 0\}.$$

With this, we get a chart from a neighborhood of $\mathbb{R}^4$ to a neighborhood of the skeleton of a $A^{(i)}_{E-X^2}(M)$ near any $A^{(j)}_{E^3}$.

The last thing left is to examine the gluing of $FF-E$ points to $E-X^2$. Here, after we have left-composed by the diffeomorphism G^{-1}_{FF-E}, the local coordinates $(z_1, z_2, x^e, y^e) \in \mathbb{C}^2 \times \mathbb{R}^2 \simeq \mathbb{R}^6$ near a $A^{(l)}_{FF-E}(M)$ are the same, except that now we look at one of the $A^{(i)}_{E-X^2} \subseteq \mathrm{CrP}_{E-X^2}$, and $A^{(l)}_{FF-E} \subseteq \mathrm{CrP}_{FF-E}$. We have

$$A^{(i)}_{X^2-E} = \{(z_1 = x_1 + ix_2, z_2 = \xi_1 + i\xi_2, 0, 0) \mid z_1, z_2 \neq 0\}.$$

In the same coordinates, the $FF-E$ point is

$$A^{(l)}_{FF-E} = (0,0,0,0).$$

Thus, the skeleton near a $FF-E$ point is

$$A^{(l)}_{FF-E} \cup A^{(i)}_{E-X^2} = \{(z_1, z_2, 0, 0) \mid z_1, z_2 \in \mathbb{C}\}.$$

That is, we get another chart from a neighborhood of $\mathbb{R}^4$ to a neighborhood of the skeleton of a $A^{(i)}_{E-X^2}(M)$ near $A^{(l)}_{FF-E}$.

If we look at the skeleton of a given $A^{(i)}_{\Bbbk}$, the symplectic form $\tilde{\omega}^{(i)}_{\Bbbk}$ is the one given by $A^{(i)}_{\Bbbk}$. This completes the proof of the manifold part of Theorem 4.1.4 for $2n = 6$.

Before proving Theorem 4.1.4 for every dimension, we first prove the following lemma:

Lemma 4.1.9. *Let $(M^{2n}, \omega, F = (f_1, \ldots, f_n))$ be an semi-toric integrable Hamiltonian system with M compact. Then, we have*

$$\mathrm{depth}(M, \mathbf{CrP(F)}) = \sup_{\Bbbk \in \mathcal{W}^n_0(F)} \mathrm{CrP}^F_{\Bbbk}(M) \leqslant n \ :$$

the maximal possible depth for a stratification induced by the $\mathrm{CrP}^F_{\Bbbk}(M)$ is the number of degrees of freedom. It is attained if and only if the integrable Hamiltonian system has a fixed point of Williamson type E^n.

Proof. Given the partial order on $\mathcal{W}^n_0(F) \subseteq \mathcal{W}^n_0$ and Eliasson normal form, the deepest possible point is a fixed point of Williamson type E^n. The associated chain of strata is thus

$$E^n \preccurlyeq \left(E^{n-1} - X\right) \preccurlyeq \left(E^{n-2} - X^2\right) \preccurlyeq \cdots \preccurlyeq X^n \tag{4.1}$$

Indeed, had it been a chain with a FF that realized the depth of $\mathcal{W}_0^n(F)$, with the constraint 1.6 on the coefficients: $k_{\mathrm{e}}+2k_{\mathrm{f}}+k_{\mathrm{x}}=n$, we'd have, in the worst case scenario

$$\left(FF-E^{n-2}\right) \preccurlyeq \left(FF-E^{n-3}-X\right) \preccurlyeq \cdots \preccurlyeq \left(FF-X^{n-2}\right) \preccurlyeq X^n$$

that is, a $(n-1)$-chain. Last thing is that with equation 4.1 one can see that the dimension $2n$ is actually twice the maximal depth of $\mathcal{W}_0^n(F)$.

$\square$

Now that we know that the depth of the stradispace induced by an integrable Hamiltonian system is always smaller than half its dimension, let's take $r \in \mathbb{N}$. By induction on $d = \operatorname{depth}(M) = \sup_{\Bbbk \in \mathcal{W}_0^n(F)} dp_M(\mathrm{CrP}_{\Bbbk}(M))$, we will prove the following proposition:

Proposition 4.1.10. $\mathbf{B_d}$: *$\forall r \in \mathbb{N}$, for all integrable Hamiltonian system $(M^{2r}, \omega, F = (f_1, \ldots, f_r))$ of dimension $2r$ and of depth d, all the skeleta of the induced stratification* **CrP(F)** *are submanifolds.*

$\mathbf{B_0}$**:** regardless of the dimension, if $\sup_{i\in\mathcal{I}} dp_M(\mathrm{CrP}_{\Bbbk}(M)) = 0$, all pieces are of depth 0, so they are a union of manifolds, and thus again a manifold.

$\mathbf{B_1}$**:** it is not clear as whether for all r, for every integrable Hamiltonian system $(M^{2r}, \omega, F = (f_1, \ldots, f_r))$ of dimension $2r$ and of depth $depth(M) = 1$, all the skeleta of the induced stratification **CrP(F)** are submanifolds, or if it is just because this case never happens.

$\mathbf{B_d} \implies \mathbf{B_{d+1}}$**:** we assume that $\forall r \in \mathbb{N}$, for all integrable Hamiltonian system $(M^{2r}, \omega, F = (f_1, \ldots, f_r))$ of dimension $2r$ and of depth $depth(M) \leqslant d$, all the skeleta of the induced stratification **CrP(F)** are submanifolds.

Now, we take an integrable Hamiltonian system $(N^{2r'}, \omega', F')$, of arbitrary dimension $2r' \in \mathbb{N}$, such that its induced stratification is of depth $d+1$. Let's show that all its skeleta are still submanifolds.

With lemma 4.1.9, we know that $d+1 \leqslant r'$. With Eliasson normal form, we also have that for a given $\Bbbk = (k_{\mathrm{e}}, k_{\mathrm{f}}, k_{\mathrm{x}})$, the strata immediately parent (see definition 1.3.14) to a $A_{\Bbbk}^{(i)}$ are of Williamson type $\Bbbk'$ with $k'_{\mathrm{x}} = k_{\mathrm{x}} + 1$ or $k'_{\mathrm{x}} = k_{\mathrm{x}} + 2$, and since we are dealing with semi-toric systems, in the chain there can be only one stratum which "jumps" from k_{x} to $k_{\mathrm{x}} + 2$, all the other strata move from k_{x} to $k_{\mathrm{x}} + 1$ (we already mention that fact in lemma 4.1.9). Last thing is that $\operatorname{rk}(F) = r'$ (i.e. is maximal) on an open dense subset of $N^{2r'}$.

The consequence of all these facts is that the depth of an integrable Hamiltonian system M^{2r} is always realized by a chain that starts at the – unique – connected component of $\mathrm{CrP}_{X^r}(M)$ and ends at the union of the deepest strata (regardless of the order, regular or reverse, see Example 1.3.18). Hence here, for depth $d+1$

$$\mathrm{CrP}_{X^{r'}}(N) \geq \bigcup_{i_1=1}^{N_1} A^{(i_1)}_{\Bbbk_1} \geq \ldots \geq \bigcup_{i_{d+1}=1}^{N_{d+1}} A^{(i_{d+1})}_{\Bbbk_{d+1}} \tag{4.2}$$

For all the other strata, the depth of the chains obtained are $\leqslant d$ and thus are disposed of by the induction hypothesis $\mathbf{B_d}$. Hence, it is only for the chain (4.2) that we will need to show $\mathbf{B_{d+1}}$, using $\mathbf{B_d}$ somewhere along the way. The initial stratum is of Williamson type $X^{r'}$. The terminal strata can be of Williamson type $\Bbbk_{d+1} = \Bbbk^{r'}_{\beta} = E^b - X^{r'-b}$ or of Williamson type $\Bbbk_{d+1} = \Bbbk^{r'}_{\gamma} = FF - E^c - X^{r'-c-2}$, with $b, c \geqslant 0$ and r' an arbitrary but finite number of degrees of freedom. We have

$$d+1 = depth(N) = \begin{cases} b & \text{in the first case} \\ c+2 & \text{in the second case} \end{cases}$$

Of course the definition of the depth of a stratum remains unchanged, as does the definition of the depth of a stradispace. That means in particular that $d+1 \leqslant r'$ but it is actually rather hard to get more information about the possible depth of $(M, \mathbf{CrP}(\mathbf{F}))$.

About the terminal strata, that is, the deepest strata, we can notice they are all closed. Thus what is left once all the terminal strata are removed is an integrable Hamiltonian system of dimension $2r'$, and its induced stratification is of depth d. We can therefore apply again our induction hypothesis $\mathbf{B_d}$: *all the skeleta of the induced stratification* $\mathbf{CrP}(\mathbf{F})$ *are submanifolds.* We can suppose that the skeleton of $CrP_{X^{r'}}(N)$ minus the terminal strata $\bigcup_{i_{d+1}=1}^{N_{d+1}} A^{(i_{d+1})}_{\Bbbk_{d+1}}$ is of the form

$$CrP_{X^{r'}}(N) \geq \cdots \geq \bigcup_{i_d=1}^{N_d} A^{(i_d)}_{\Bbbk_d}, \text{ with the d-th Williamson type}$$

$$\Bbbk_d = \begin{cases} \text{If } \Bbbk_{d+1} = \Bbbk^{r'}_{\beta}\ , & E^{b-1} - X^{r'-(b-1)}(k_{\mathrm{x}} \to k_{\mathrm{x}}+1) \\ \text{If } \Bbbk_{d+1} = \Bbbk^{r'}_{\gamma} & \begin{cases} FF - E^{c-1} - X^{r'-c-1}(k_{\mathrm{x}} \to k_{\mathrm{x}}+1) \\ E^c - X^{r'-c}(k_{\mathrm{x}} \to k_{\mathrm{x}}+2) \end{cases} \end{cases} .$$

We apply our induction hypothesis, and have that the union

$$\mathcal{A} = \bigcup_{l=1}^{d} \bigcup_{i_l=1}^{N_l} A^{(i_l)}_{\Bbbk_l}$$

is a submanifold. Hence, the only thing that matters now is to examine the terminal strata, to show that when we glue it back we obtain again a submanifold.

The number of possible Williamson type for the terminal strata

$$\bigcup_{i_{d+1}=1}^{N_{d+1}} A_{\Bbbk_{d+1}}^{(i_{d+1})}$$

is finite. We thus need to show that for each possible Williamson type, Eliasson normal form provides us smooth chart of a terminal stratum with the desired Williamson type.

We know there exists a symplectomorphism $\varphi_{\Bbbk_{d+1}} : (M, \omega) \to (\mathbb{R}^n, \omega_0)$ and $G_{\Bbbk_{d+1}}$ a diffeomorphism of $\mathbb{R}^n$ such that

$$\varphi_{\Bbbk_{d+1}}^* F = G_{\Bbbk_{d+1}} \circ Q_{\Bbbk_{d+1}}.$$

We can now left-compose by $G^{-1}_{\Bbbk^{r'}_{\beta}}$ (resp. $G^{-1}_{\Bbbk^{r'}_{\gamma}}$). The strata immediately parent to the terminal strata are strata of depth d. When $\Bbbk_{d+1} = \Bbbk^{r'}_{\beta}$, a terminal stratum is, in local coordinates

$$A^{(i)}_{\Bbbk^{r'}_{\beta}} = \{(x_1^e, y_1^e, \ldots, x_b^e, y_b^e) = (0, \ldots, 0) \ , \ \boldsymbol{\theta} \in \mathbb{T}^{r'-(d+1)} \ , \ \boldsymbol{\xi} \in]-\varepsilon, \varepsilon[^{r'-(d+1)}\}.$$

If we remove it, the immediate parent strata to the terminal stratum are one of the N_d possible $A^{(i_d)}_{\Bbbk_d}$ that appears in the union $\bigcup_{i_d=1}^{N_d} A^{(i_d)}_{\Bbbk_d}$, with

$$A^{(i_d)}_{\Bbbk_d} = \{x_j^e = y_j^e = 0 \ , \ j \neq i_d \ , \ q^e_{i_d} > 0 \text{ and } \boldsymbol{\theta} \in \mathbb{T}^{r'-(d+1)} \ , \ \boldsymbol{\xi} \in]-\varepsilon, \varepsilon[^{r'-(d+1)}\}.$$

The $A^{(i_d)}_{\Bbbk_d}$ are strata of depth d, and the induction hypothesis tells us that

$$\mathcal{A} = \bigcup_{l=1}^{d} \bigcup_{i_l=1}^{N_l} \tilde{A}^{(i_l)}_{\Bbbk_l} = \left\{ q_j^e > 0 \ , \ j = 1..d+1 \ , \ \boldsymbol{\theta} \in \mathbb{T}^{r'-(d+1)} \ , \ \boldsymbol{\xi} \in]-\varepsilon, \varepsilon[^{r'-(d+1)} \right\}.$$

is a submanifold. We then glue back the terminal stratum $A^{(i)}_{\Bbbk^{r'}_{\beta}}$ to $\mathcal{A}$. What we get is diffeomorphic to $\mathcal{B}^{2r'}(0, \varepsilon)$, that is, a submanifold of dimension $2r'$.

Next, if $\Bbbk = \Bbbk^{r'}_{\gamma} = FF - E^c - X^{r'-c-2}$ we have, for $j = 1 \ldots d+1$, in local cooordinates

$$\begin{aligned} &A^{(0)}_{\Bbbk^{r'}_{\gamma}} = \\ &\Big\{ (z_1 = x_1 + ix_2, z_2 = \xi_1 + i\xi_2, \mathbf{x}^e, \mathbf{y}^e) \mid z_1 = z_2 = 0 \ , \\ &\quad q^e_l = 0 \ , l = 1 \ldots c \text{ and } \boldsymbol{\theta} \in \mathbb{T}^{r'-c} \ , \ \boldsymbol{\xi} \in]-\varepsilon, \varepsilon[^{r'-c} \Big\}. \end{aligned}$$

There are now two possibilities:

– The stratum immediately parent is $A^{(1)}_{E^{d-1}-X^{r'-(d-1)}}$: it means we moved away from a focus-focus component. As a consequence, the rank of the stratum "jumped" from k_{x} to $k_{\mathrm{x}} + 2$, and the next parent strata won't have focus-focus component.

– Or, the strata immediately parent are one of the N'_d strata $A^{(j)}_{\Bbbk_\delta}$ that appear in the union $\bigcup_{i'_d=1}^{N'_d} A^{(i'_d)}_{\Bbbk^{r'}_\delta}$ where $\Bbbk^{r'}_\delta = FF - E^{d-2} - X^{r'-d}$.

We shall thus consider the union

$$\left(A^{(1)}_{E^{d-1}-X^{r'-(d-1)}}\right) \cup \left(\bigcup_{j=1}^{d+1} A^{(j)}_{\Bbbk_\delta}\right).$$

of strata immediately parent to $A^{(i)}_{\Bbbk_\delta}$. We have, in the $\Bbbk^{r'}_\gamma$ local coordinates,

$$A^{(1)}_{E^{d-1}-X^{r'-(d-1)}} = \left\{ \begin{array}{c} (z_1, z_2, \mathbf{x}, \boldsymbol{\xi}) \mid z_1, z_2 \neq 0 \ , \ (\mathbf{x}^e, \mathbf{y}^e) = (0,0) \\ \text{and } \boldsymbol{\theta} \in \mathbb{T}^{r'-(d+1)} \ , \ \boldsymbol{\xi} \in]-\varepsilon, \varepsilon[^{r'-(d+1)} \end{array} \right\}.$$

and, for $j = 1 \ldots d-1$,

$$A^{(j)}_{\Bbbk^{r'}_\delta} = \begin{array}{l} \Big\{ (z_1, z_2, \mathbf{x}, \boldsymbol{\xi}) \mid z_1, z_2 = 0 \ , \ q^e_i = 0 \ i \neq j \ q^e_j > 0 \\ \qquad \text{and } \boldsymbol{\theta} \in \mathbb{T}^{r'-(d+1)} \ , \ \boldsymbol{\xi} \in]-\varepsilon, \varepsilon[^{r'-(d+1)} \Big\}. \end{array}$$

Either way, we now appply our induction hypothesis the same way we did in the $E^{d+1} - X^{r'-(d+1)}$ case. We have that

$$\begin{aligned} \mathcal{B} &= \left(\tilde{A}^{(1)}_{E^{d-1}-X^{r'-(d-1)}}\right) \cup \left(\bigcup_{j=1}^{d+1} \tilde{A}^{(j)}_{\Bbbk_\delta}\right) \\ &= \Big\{ z_1, z_2 \neq 0 \ , \ q^{\mathrm{e}}_l = 0 \ , \ l = 1..d+1 \ , \ \boldsymbol{\theta} \in \mathbb{T}^{r'-(d+1)} \ , \ \boldsymbol{\xi} \in]-\varepsilon, \varepsilon[^{r'-(d+1)} \Big\} \\ &\cup \Big\{ z_1 = z_2 = 0 \ , \ q^{\mathrm{e}}_l > 0 \ , \ l = 1..d+1 \ , \ \boldsymbol{\theta} \in \mathbb{T}^{r'-(d+1)} \ , \ \boldsymbol{\xi} \in]-\varepsilon, \varepsilon[^{r'-(d+1)} \Big\} \end{aligned}$$

is a submanifold.

Now, we glue back $A^{(0)}_{FF-E^{d-1}-X^{r'-(d+1)}}$ to the submanifold $\mathcal{B}$. We have a chart from a neighborhood of the terminal stratum $A^{(0)}_{FF-E^{d-1}-X^{r'-(d+1)}}$ to a neighborhood of $\mathcal{B}^{2r'}(0, \varepsilon)$.

We have thus proved that in all the possible cases that can occur, in the passage of depth d to depth $d+1$ we keep on having manifolds of dimension $2k_{\mathrm{x}}$, and k_{x} can take its values between 0 and r', but r' is arbitrary. We keep on taking the union of manifolds, and hence, we have again manifolds. Remember that the cases of rank $k_{\mathrm{x}} < r'$ have been treated by the induction hypothesis.

Given an integrable Hamiltonian system and an initial connected component of a stratum $A^{(0)}_{\Bbbk_0}$, we have constructed the skeleta of this $A^{(0)}_{\Bbbk_0}$, by taking the union of all strata $\leq A^{(0)}_{\Bbbk_0}$. We have proved with Elliasson normal form that it is a manifold, of dimension $2k_{\mathrm{x}}$.

Now, the stratum of dimension $2k_{\mathrm{x}}$ of the stratification actually have a symplectic structure. Since the skeleton is just the maximal stratum plus symplectic submanifolds of strictly smaller dimension, this amounts to extend the symplectic structure of the stratum of maximal dimension to the whole skeleton.

In Theorem 3.1.11 we have, for a leaf Λ of Williamson type $\Bbbk$, an effective Hamiltonian action of $\mathbb{T}^{k_{\mathrm{f}}+k_{\mathrm{e}}+k_{\mathrm{x}}}$ on a tubular neighborhood $\mathcal{V}(\Lambda)$ of Λ. The $\mathbb{T}^{n-1}$-action given by $\check{F}$ in the definition of a semi-toric system is the action of a sub-torus of this $\mathbb{T}^{k_{\mathrm{f}}+k_{\mathrm{e}}+k_{\mathrm{x}}}$, strict in general (think of the case of a regular leaf). We can restrict the action of F to a stable subset like the $\tilde{A}^{(i)}_{\Bbbk}$, which we have proven to be also a symplectic manifold. We have that $F^{(i)}_{\Bbbk}$ is in $\mathcal{C}^{\infty}(\tilde{A}^{(i)}_{\Bbbk} \to \mathbb{R}^n)$ and since $i_{\tilde{A}^{(i)}_{\Bbbk}}$ is a symplectic embedding, we can have that $(\tilde{A}^{(i)}_{\Bbbk}, \tilde{\omega}^{(i)}_{\Bbbk}, F^{(i)}_{\Bbbk})$ is what is usualy called a superintegrable system by the Russian school (see [MF78],[Fas05], [BJ03]), that is, a Hamiltonian system on a $2k_{\mathrm{x}}$-dimensional symplectic manifold for which:

$\triangleright$ There exist $n \geqslant k_{\mathrm{x}}$ independent integrals f_i of motion. Their level surfaces (invariant submanifolds) form a fibered manifold $F : Z \to N = F(Z)$ over a connected open subset $N \subset \mathbb{R}^k$.

The case $k = n$ is simply the Liouville integrable case. Now, we know that F is semi-toric and $\tilde{A}^{(i)}_{\Bbbk}$ is stable for the flow of F. In dimension $2n = 6$, we have two possible scenarii when we restrict the moment map to a skeleton $\tilde{A}^{(i)}_{\Bbbk}$ of dimension 4: this is what we shall treat here.

- First, the $\mathbb{T}^2$-action can be locally free on $\tilde{A}^{(i)}_{\Bbbk}$: that means the rank of (df_2, df_3) on a skeleton is 2 on a dense set of $\tilde{A}^{(i)}_{\Bbbk}$. Then, we still have a moment map for a Hamiltonian $\mathbb{T}^2$-action on the skeleton : we have a semi-toric system.
- We can also have that $\mathrm{rk}(f_2, f_3) \leqslant 1$ on an open set $\mathcal{U}$. Note that in this case the rank cannot be 0 on $\mathcal{U}$: indeed, we took the skeleton of a stratum of the form $A^{(i)}_{E-X^2}$, hence $\mathrm{rk}(F) = 2$ almost everywhere, and thus $\mathrm{rk}(f_2, f_3) \geqslant 1$, that is, almost everywhere on the skeleton. Therefore, $\mathrm{rk}(df_2, df_3) = 1$ almost everywhere on $\mathcal{U}$: we have a Hamiltonian S^1-action of rank 1 on $\mathcal{U}$. That means we can find a linear combination $af_2 + bf_3$ with integer coefficients almost everywhere on $\mathcal{U}$. This linear combination can be extended to all the skeleton but for a set of measure zero. Indeed, if we suppose the contrary, that is, that we cannot extend $af_2 + bf_3$ almost everywhere on the skeleton, that means that we are

forced to change coefficients in the linear combination af_2+bf_3 on a Cantor set. This contradicts our previous hypothesis that $\mathrm{rk}(df_2, df_3) = 1$. We then consider f_1: it is a good candidate for the first component of our semi-toric moment map. Indeed, it is only if the skeleton is the stratum $\{df_1 = 0\}$ that f_1 may degenerate by being constant. But even in this case, we'd just take (f_2, f_3) as components of the moment map.

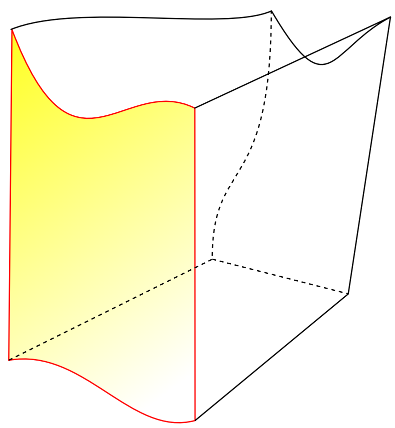

Figure 4.2: Example of a vertical stratum, here in yellow. The $E-E-X$ paths are in red. Here $k_{\mathrm{x}} \neq \check{k}_{\mathrm{x}}^{(i)}$.

So, in all cases, having at least an S^1-action implies having a linear combination of f_2 and f_3 with integer coefficients that we can then extend (almost everywhere) on the skeleton: we always have a semi-toric integrable Hamiltonian System on $\tilde{A}_{\mathbb{k}}^{(i)}$.

We say a few more words about the case of arbitrary dimension. We still have that $\check{F}_{\mathbb{k}}^{(i)} := \pi_\Delta \circ F \circ i_{\tilde{A}_{\mathbb{k}}^{(i)}}$ induces an action of $\mathbb{T}^{n-1}$, but the rank $\check{k}_{\mathrm{x}}^{(i)}$ of $\check{F}_{\mathbb{k}}^{(i)}$ is not equal to k_{x} in general. However, since $k_{\mathrm{x}} = n - dp_M(A_{\mathbb{k}}^{(i)})$, the difference $\Delta k_{\mathrm{x}} = k_{\mathrm{x}} - \check{k}_{\mathrm{x}}^{(i)}$ is at most 1: it is because we have a semi-toric system. Thus, with arguments similar to the case $2n = 6$, we should be able to prove that there exists a $A \in GL_{\check{k}_{\mathrm{x}}^{(i)},n-1}(\mathbb{Z})$ such that

$$A \circ \check{F}_{\mathbb{k}}^{(i)}$$

gives a Hamiltonian action of a torus $\mathbf{T}_{\Bbbk}^{(i)}$ on $\tilde{A}_{\Bbbk}^{(i)}$ of rank $\check{k}_{\mathrm{x}}^{(i)}$. It is important to understand that *the difference between* k_{x} *and* $\check{k}_{\mathrm{x}}^{(i)}$ *is not related to the presence of focus-focus singularities in the skeleton*, for the same reason that a semi-toric system on a symplectic manifold M^{2n} may actually have a full $\mathbb{T}^n$-action.

□

Let's show now the following lemma:

Lemma 4.1.11. *If* (M', ω', G) *is a semi-toric integrable system, for regular values of* $\check{G}$*,* g_1^+ *and* g_1^- *are well defined and are smooth.*

Proof. Let $p \in \Delta_G$ be a regular value of $\check{G}$ (remember that being regular is an open condition). The fiber $\check{G}^{-1}(p)$ is connected, and it is a smooth compact manifold as p is regular. The function $g_1|_{\check{G}^{-1}(p)}$ is Morse-Bott, thus by Morse-Bott theorem the locus C^+ of local maxima and C^- of local minima are connected: all local maxima/minima are global ones. Thus $g_1^{\pm}$ depends only of p. Since here at least in a small open neighborhood of p the fibers $\check{G}^{-1}(p)$ are homotopic, we can use a parametrized version of Morse-Bott theorem to prove that $g_1^{\pm}$ depends smoothly of p.

□

The index, that is, the number of negative eigenvalues of the Hessian, is locally constant along the critical submanifold. Now we take the following proposition:

Proposition 4.1.12. $\underline{\mathbf{P_n}:}$ *The* **Description** *Theorem 4.1.7 is true for all semi-toric systems of dimension less than or equal to* $2n$*.*

and use it to prove Theorem 4.1.7 for all semi-toric integrable systems by induction.

Proof of Theorem. 4.1.7

$\underline{\star)\mathbf{P_2}:}$ It has already been proven; this is the content of Proposition **2.9.** and Theorem **3.4** in [VN07].

We will show the induction relation for the max function $M_{\Bbbk}^{(i)}$, as it works exactly the same for the min function $m_{\Bbbk}^{(i)}$, *mutatis mutandis.*

$\underline{\star)\mathbf{P_n} \Rightarrow \mathbf{P_{n+1}}:}$

We suppose that P_n is true, and we consider a semi-toric integrable system of $n+1$ degrees of freedom (M, ω, F), with $F = (f_1, \ldots, f_{n+1})$, with $\check{F}$ giving a global Hamiltonian action of $\mathbb{T}^n$. Items 1) to 4) are still valid, we use the same notations, and all skeleta $\tilde{A}_{\Bbbk}^{i)}$ of connected components of CrP's except for the whole $F(M)$ which has $n+1$ degrees of freedom yield semi-toric integrable systems with degrees of freedom $\leqslant n$, so we can already apply the induction hypothesis to all these skeleta.

We first prove the **Description** item **a.**. If we have $k_{\mathrm{x}}^{(i)} = \check{k}_{\mathrm{x}}^{(i)}$, this means that $(d(f_1)|_{A_{\Bbbk}^{(i)}})_p = 0$ for all $p \in A_{\Bbbk}^{(i)}$, so $(F_{\Bbbk}^{(i)})_1^+ = (F_{\Bbbk}^{(i)})_1^- = (F_{\Bbbk}^{(i)})_1$, and $(F_{\Bbbk}^{(i)})_1$ is a smooth function. Note that here, the only use of the induction hypothesis we made was the disjunction of cases. The disjunction of cases can be reformulated with the following lemma

Lemma 4.1.13. *If (M', ω', G) is a semi-toric integrable system, either $k_{\mathrm{x}}^{(i)} = \check{k}_{\mathrm{x}}^{(i)}$ or $k_{\mathrm{x}}^{(i)} = \check{k}_{\mathrm{x}}^{(i)} + 1$.*

Proof. The proof is part of the induction hypothesis for all strata except for $\mathrm{CrP}_{X^n}(M)$, but for this stratum, it is clear that $k_{\mathrm{x}}^{(i)} - \check{k}_{\mathrm{x}}^{(i)} = 1$. □

Note that this result also proves that a vertical tangency in the image of the moment map can only occur at the frontier of a $A_{\Bbbk}^{(i)}$.

Now, we have the result on every strata but $\mathrm{CrP}_{X^n}(M)$. This can seem of little interest to prove $\mathbf{P_{n+1}}$ since we avoided the stratum of dimension $2(n+1)$. Yet, this leaves us only with the continuity of f_1^+ to show.

Lemma 4.1.14. *If there exists a point of discontinuity of f_1^+, then there exists an open subpolytope $\Delta_{\Bbbk}^{(i)}$ of dimension $k_{\mathrm{x}}^{(i)} \leqslant n$ of Δ, image by $\check{F}$ of a stratum $A_{\Bbbk}^{(i)}$ with semi-toric skeleton $(\tilde{A}_{\Bbbk}^{(i)}, \tilde{\omega}_{\Bbbk}^{(i)}, F_{\Bbbk}^{(i)})$ and with $B_{\Bbbk}^{(i)}$ for image by F, such that $k_{\mathrm{x}}^{(i)} \neq \check{k}_{\mathrm{x}}^{(i)}$, and such that for all $\check{q} \in \Delta_{\Bbbk}^{(i)}$, f_1^+ is discontinuous in $\check{q}$.*

Proof. Let $\check{p}$ be a point of discontinuity of f_1^+. The point $r = (f_1^+(\check{p}), \check{p})$ is in a $B_{\Bbbk'}^{(i')}$, image of a stratum $A_{\Bbbk'}^{(i')}$ by F. Since it is in $\partial F(M)$, we necessarily have that $k'_{\mathrm{x}} < n+1$, and it can't be equal to n either (we'd have $k_{\mathrm{x}} = \check{k}'_{\mathrm{x}}$ and $f_1^+ = (F_{\Bbbk'}^{(i)})_1^+$ locally around $\check{p}$, and we know $(F_{\Bbbk'}^{(i)})_1^+$ is a smooth function of n variables). Thus, $k'_{\mathrm{x}} \leqslant n-1$.

A consequence of Theorem 3.1.14 is that, since f_1^+ is discontinuous in $\check{p}$ and $F(M)$ is closed, there exists a $B_{\Bbbk}^{(i)}$ such that $B_{\Bbbk'}^{(i)} \leq B_{\Bbbk}^{(i)}$, with $B_{\Bbbk}^{(i)}$ vertical with respect to π_Δ. So $k_{\mathrm{x}}^{(i)} = \check{k}_{\mathrm{x}}^{(i)} + 1 = k'_{\mathrm{x}} + 1 \leqslant n$. Therefore, $\Delta_{\Bbbk}^{(i)} = \Delta_{\Bbbk'}^{(i')}$, and we have that f_1^+ is discontinuous for all $\check{q} \in \Delta_{\Bbbk}^{(i)}$.

□

Now, let's examine more cautiously the $B_{\Bbbk}^{(i)}$ given by Lemma 4.1.14. We know that it is vertical with respect to π_Δ and that f_1^+ is discontinuous for all $\check{q} \in \Delta_{\Bbbk}^{(i)}$. We now prove the following lemma:

Lemma 4.1.15. *The function f_1^+ is also discontinuous at the frontier $\partial\Delta_{\Bbbk}^{(i)}$.*

Proof. If we suppose otherwise, we get Eliasson local models that are not possible. We prove it here with the figure below for dimension $2n = 6$, but it works exactly the same in higher dimension. We'd get something like the figure below, with edges corresponding to $B^{(i)}_{E^2-X^{n-2}}(M)$ and the vertices we circled, that correspond to no local model.

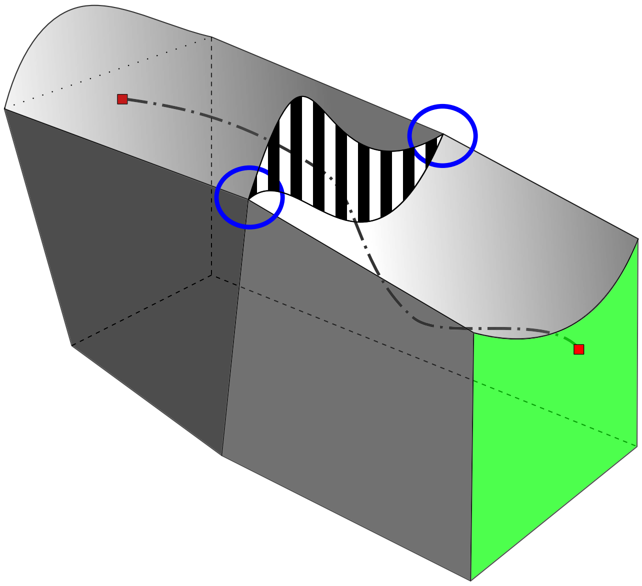

Figure 4.3: Blue circles indicate Eliasson normal forms that can't happen with non-degenerate semi-toric Hamiltonian systems (here in dimension 6)

$\square$

So this means that there is a vertical $B^{(i)}_{\Bbbk}$ of dimension n with a discontinuity of f_1^+. It corresponds to a semi-toric skeleton $(\tilde{A}^{(i)}_{\Bbbk}, \tilde{\omega}^{(i)}_{\Bbbk}, F^{(i)}_{\Bbbk})$. But this contradicts our induction hypothesis.

So now we have continuity of f_1^+ and f_1^-, and this proves $\mathbf{P_{n+1}}$. The fibration property comes immediately from it, and hence this concludes the proof of Theorem 4.1.7.

$\square$

We can notice that, when applied to the case $2n = 6$ and $\Bbbk = FF - X$, Description item **a.** of Theorem 4.1.7 gives another proof of Theorem 3.2.10.

Here we have that $0 \leqslant k_{\mathrm{x}} - \check{k}^{(i)}_{\mathrm{x}} \leqslant 1$ because we are in the case of complexity one. A generalization of this theorem to higher complexity, though probably much harder to formulate for the description part, should still be possible.

There is a lot of information one can deduce of all the results of Section 4.1.1 about what can and what cannot occur for the image of the moment map of a semi-toric integrable Hamiltonian system. For instance, since in the Serre fibration of $F(M)$ the base and the fiber have trivial homotopy groups we have the immediate corollary

Corollary 4.1.16. *The image of the moment map is contractible.*

4.2 Connectedness of the fibers

4.2.1 Morse theory

The proof of the convexity theorem given by Atiyah relies heavily on Morse theory. We will recall here some of the results we use from this theory that we will use to prove our own fiber-connectedness result.

Theorem 4.2.1 (Sard)**.** *Let f be a function in $\mathcal{C}^\infty(M \to N)$. The set of critical values of a differentiable function is of zero measure.*

Definition 4.2.2. *A function $f \in \mathcal{C}^\infty(M \to \mathbb{R})$ is Morse-Bott if its critical set $\mathrm{Crit}(f) = \{x \in M|\ df(x) = 0\}$ decomposes into finitely many closed connected submanifolds, which we shall call the critical manifolds, and if for a critical point p we have:*

$$\ker(\mathcal{H}(f)_p) = T_p\mathrm{Crit}(f).$$

That is, we ask that the Hessian of a Morse-Bott function to be transversally non-degenerate. We define the index i_+ and the co-index i_- of f at p as the number of negative eigenvalues of $\mathcal{H}(f)_p$ and $\mathcal{H}(-f)_p$ respectively.

Morse theory is based on these functions. Hopefully, the set of Morse-Bott functions is dense is the set of smooth functions, so that we can always perturbate a smooth function so that it becomes Morse-Bott. A lot of Morse theory deals with how the level sets "glue" with each other, but our concern is in the connectedness of the fibers so we'll focus on the following theorem

Theorem 4.2.3 (Morse fiber-connectedness)**.** *Let M be a compact, connected manifold, and f a Morse-Bott function with indexes and co-indexes are $\neq 1$.*

Then we have the following assertions:

- *The set of local minima (resp. maxima) is connected. Its image is unique, and thus it is the globally minimal (resp. maximal) value of f.*
- *The fibers of f are connected.*

Proof. **Claim (1):** Reductio ad absurdum

Let x, y be two local minima each belonging to distinct connected components. Since M is a connected manifold, it is path-connected: we can construct a smooth path $\gamma : [0, 1] \to M$ from x to y. We can always choose γ such that $|\gamma'|(t) \neq 0$. We thus have $f \circ \gamma$ as a function from $[0, 1]$ to $\mathbb{R}$, with 0 and 1 as local minima. Hence there exists by Rolle theorem a $t_\gamma \in]0, 1[$ such that $f \circ \gamma(t_\gamma) = \max_{[0,1]} f \circ \gamma(t)$. We set $z[\gamma] := \gamma(t_\gamma)$.

By construction, a $z[\gamma]$ is on $\mathrm{Crit}(f)$:

$$(f \circ \gamma)'(t_\gamma) = 0 = df_{z[\gamma]}.\underbrace{\gamma'(t_\gamma)}_{\neq 0}$$

Remind that since f is Morse-Bott, $\mathrm{Crit}(f)$ is a disjoint union of closed connected submanifolds. As we only need to prove that there is at least one critical point with index or co-index 1, we can only take $z[\gamma]$'s that come from paths that intersect transversally each connected components of $\mathrm{Crit}(f)$: it is always possible in the smooth category. We can also take only one $z[\gamma]$ for each connected component of $\mathrm{Crit}(f)$ intersected by the γ's. This gives us a set Γ of acceptable paths and a set Z of points $z[\gamma]$ of M.

Now, we take z_0 such that: $f(z_0) = \min_{z[\gamma]\in Z} f(z[\gamma])$.

Let's show its index is 1. There is at least one negative eigenvalue of $\mathcal{H}(f)_{z_0}$ in the direction of $\gamma'(t_\gamma)$, since it is a local maximum of $f \circ \gamma_0$. If we suppose there is another negative eignevalue, writing the Morse lemma for f with a basis adapted to γ (this is possible since γ intersect transversally $\mathrm{Crit}(f)$), taking x_1 for the direction of γ and x_2 for the other eigenvalue, we have a diffeomorphism φ from a neighborhood $\mathcal{U}$ of p to a open ball $\mathcal{B}^n(0,\varepsilon)$

$$f \circ \varphi^{-1}(x) = -x_1^2 - x_2^2 + \sum_{j=3}^{d} \epsilon_j x_j^2 \text{ where } d = n - T_{z[\gamma]}\mathrm{Crit}(f)$$

In these local coordinates, γ is a straight line directed by x_1. We can locally deform γ smoothly to a path γ' that crosses transversally the line directed by x_2. We have that $\gamma' \in \Gamma$ and that $f(z[\gamma']) < f(z[\gamma])$. This contradicts the definition of z_0. Thus, $\mathcal{H}(f)_{z[\gamma_0]}$ has exactly one negative eigenvalue.

This contradicts the hypothesis that f has indexes and co-indexes $\neq 1$. We show the same result for the local maxima since $-f$ is another Morse-Bott function with index and co-index $\neq 1$.

Claim (2):

Let c be a value of f such that $f^{-1}(c)$ is disconnected. We have to distinguish two cases:

- If both M_c^- and M_c^+ are connected:
  We have that $f^{-1}(c) = \bigsqcup_{1\leqslant i\leqslant N} A_i$ with $A_i = \partial B_i$: each connected component A_i is the border of an n-complex.
  Thus, $f^{-1} = \bigsqcup_{1\leqslant i\leqslant N} \partial B_i$, so $\partial f^{-1}(c) = 0$ but there exists no n-complex B such that $f^{-1}(c) = \partial B$: the fiber of c is a non-trivial $n-1$-cycle.
  So with Morse inequalities, posing: $C_{i_+,i_-} := \# \, of \; c.c. \; of \; \mathrm{Crit}(f)_{i_+,i_-}$, we have $C_{i_+,1} \geqslant b_{n-1} \geqslant 1$. That implies the existence of a critical point of co-index 1: contradiction.
- If M_c^- or M_c^+ is disconnected, then it implies the existence of at least two local maxima or two local minima, each being on a distinct connected component. This contradicts the claim (1).

□

4.2.2 The result

Theorem 4.2.4.
Let (M^6, ω, F) be a semi-toric system over a 6-dimensional manifold. The fibers of F are connected.

Proof of Theorem. 4.2.4
Let's note

$$\mathcal{S} = \{c \in F(M) | F^{-1}(c) \text{ is not connected }\} \text{ and } \mathcal{S}_\Delta = \pi_\Delta(\mathcal{S}).$$

Our goal here is to prove that $\mathcal{S} = \emptyset$:

1) First we suppose that $\check{c}$ is a regular value of $\check{F}$. For a given $\check{c}$ we pose $N = \check{F}^{-1}(\check{c})$. Then the following fact is true:

Lemma 4.2.5.
$(f_1)_{|N}$ is a Morse-Bott function with indices and co-indices $\neq 1$.

Proof. Since $\check{c}$ is a regular value of $\check{F}$, N is a $(n+1)$-dimensional manifold, and for all $p \in N$ we have $df_2 \wedge \cdots \wedge df_n(p) \neq 0$ $(*)$. Saying that p is a critical point of $(f_1)_{|N}$ implies, with Lagrange multipliers, that $df_1 = a_2 df_2 + \cdots + a_n df_n$ with $\vec{a} \in \mathbb{R}^{n-1}$: p is a critical point of F. It can only be of corank 1, otherwise we'd have **two** non-trivial relations between the f_i's, and this would contradict $(*)$. Thus the only possibility is that p is a critical point of Williamson type $\Bbbk = (1, 0, 0, n-1) = E - X^{n-1}$. With Miranda-Zung theorem, we have a symplectomorphism φ from an open neighborhood $\mathcal{U} \subset \mathbb{R}^2_{(x_1,y_1)} \times (T^*\mathbb{T}^{n-1})_{(\theta_2,\xi_2,\ldots,\theta_n,\xi_n)}$ to a open neighborhood of p saturated with respect to the compact orbits of p near F, sending 0 to p and $\{q_e = 0\}$ to the critical manifold, and a local diffeomorphism G of $\mathbb{R}^n$ such that:

$$(F - F(p)) \circ \varphi = G \circ Q_{E-X^{n-1}} \text{ where } Q_{E-X^{n-1}} = (q^e_{(1)} = x_1^2 + y_1^2, \xi_2, \ldots, \xi_n).$$

The question is now to use this result to get a normal form of $(f_1)_{|N}$. We know that $\check{F}^{-1}(\check{c}) = F^{-1}(I_{\check{c}})$. Using the local model near p, the equation $(*)$ implies that the intersection of N and the critical locus $\{q_e^{(1)} = 0\}$ of f_1 is transversal: thus the whole torus $\{q_e^{(1)} = 0, \xi_2 = \cdots = \xi_n = 0\} \simeq \mathbb{T}^{n-1}$ is a critical locus for $(f_1)_{|N}$. We have shown that $\mathrm{Crit}((f_1)_{|N})$ is thus a disjoint union of connected submanifold.

It is an elliptic Hessian on its transverse components, and hence it's non-degenerate. Moreover, since $\mathcal{H}(f)_p = \pm(x_1^2 + y_1^2)$, its indexes and co-indexes are either 0 or 2.

∎

Now, applying Morse's Theorem 4.2.3, we obtain that for all $\check{c}$ regular values of $\check{F}$, $(f_1)^{-1}_{|N}(c_1) = \{f_1 = c_1, \check{F} = \check{c}\} = F^{-1}(c)$ is connected. Note that this is true even if c is a critical value of F. Since we are dealing with an integrable Hamiltonian system, we know that $\mathcal{S}_\Delta$ is of empty interior in Δ, and so is $\mathcal{S}$ in $F(M)$. With Sard's theorem, we also know it is of Lebesgue measure zero. Actually, we have with Atiyah – Guillemin & Sternberg theorem that in Δ the critical values of $\check{F}$ is a disjoint union of subpolytopes. Hence, the set $\mathcal{S}_\Delta$ must be in it.

Next, we show the following lemma

Lemma 4.2.6. *Let $c \in \mathcal{S}$. Let $A_i, i = 1, \ldots, N$ be the connected components of $F^{-1}(c)$:*

$$F^{-1}(c) = \bigsqcup_{i=1}^{N} A_i.$$

For each point in A_i, $k_\mathrm{e} \geqslant 1$. In particular, there are no regular points in $F^{-1}(c)$.

Proof. Since F is proper, the A_i's are compact. If all the points of A_i are regular, A_i is a regular leaf, and we can apply Liouville-Arnold-Mineur theorem to a saturated neighborhood U_i of A_i. If there is a critical point $p \in A_i$, it is non-degenerate. We can apply Theorem 3.1.11 near p. Either way we obtain, since the system is semi-toric, that the image of an orbit-saturated neighborhood U of $p \in A_i$ is diffeomorphic to

$$[0, \varepsilon[^{k_\mathrm{e}} \times \mathcal{B}^{2k_\mathrm{f}}(0, \eta) \times \mathcal{B}^{k_\mathrm{x}}(0, \mu) \quad \text{, with } k_\mathrm{e} + 2k_\mathrm{f} + k_\mathrm{x} = n.$$

For a $A_{i'}$, $i' \neq i$, we can do the same thing and get that the image of an orbit-saturated neighborhood U' of $p' \in A_{i'}$ is diffeomorphic to

$$[0, \varepsilon[^{k'_\mathrm{e}} \times \mathcal{B}^{2k'_\mathrm{f}}(0, \eta) \times \mathcal{B}^{k'_\mathrm{x}}(0, \mu) \quad \text{, with } k'_\mathrm{e} + 2k'_\mathrm{f} + k'_\mathrm{x} = n$$

If for a point p there is no elliptic component ($k_\mathrm{e} = 0$) then $F(U)$ contains a n-dimensional ball. Thus, the volume of $F(U) \cap F(U')$ is non-zero. Now with Zung's theorem we know we can always take U and U' such that $U \cap U' = \emptyset$: the leaves of U and U' are not connected. So that means that we'd have $F(U) \cap F(U') \subseteq \mathcal{S}$ which would then be of positive measure. This is impossible.

That means that all the points in $F^{-1}(c)$ are critical points for F with at least one elliptic component.

□

Now we show that $\mathcal{S} \subseteq \partial\Delta$, that is, that all values in the interior of $\pi_\Delta^{-1}(\Delta)$ have connected fibers.

Let's take $c \in \mathcal{S}$ such that $\pi_\Delta \subseteq \Delta$: even if there are leaves with focus-focus components, the previous lemma tells us there must be also elliptic components. With the local models given by Miranda-Zung, we see that there must be another value in the vicinity whose fiber is composed of two leaves A and

B of Williamson type $E - X^{n-1}$. Hence, these values are in $\mathcal{S}$. With the local models again, we have that these values come as $(n-1)$-parameter families: immersed hypersurfaces $\mathcal{H}$. More precisely, they are affine hyperplanes that intersect $F(M)$. We will show that such hypersurfaces, such hyperplanes cannot exist.

First, there is a *verticality* constraint for $\mathcal{H}$: since $\mathcal{H}$ is a set of disconnected values, and the regular values are connected, $\pi_\Delta(\mathcal{H})$ must be of zero measure (it is an immersed $(n-2)$-surface in Δ). This verticality constraint plus the knowledge of the local model near a $E - X^{n-1}$ critical point implies that for all $c \in I_{\check{c}}$, the fibers $F^{-1}(c)$ are disconnected. Thus $F^{-1}(I_{\check{c}}) = \check{F}^{-1}(\check{c})$ is disconnected. Impossible, by A. – G. & S. theorem.

So now since $F(M)$ is closed the only possibility for $c \in \mathcal{S}$ is to be on one of the skeleta of the frontier of $F(M)$ but the frontier of the image of a semi-toric system is again the image of a semi-toric system (we proved it in dimension 3, see Section 4.1.1). San Vũ Ngọc already showed in dimension $2n = 4$ that the fibers of a semi-toric integrable system were connected. Thus, an immediate induction over n gives us that $c \in \mathcal{S}$ cannot exist: the fibers of a semi-toric integrable system, in any dimension, are hence always connected.

We give another proof that there cannot be disconnected fibers on $\partial F(M)$: $F^{-1}(c) = \sqcup_{i=1}^{N} A_i$. For $c \in \mathcal{S} \cap \partial F(M)$, let

$$j_c = \max \left\{ k \in \mathbb{N} \mid \exists \varepsilon > 0 \ s.t.\ \mathcal{B}^k(c, \varepsilon) \hookrightarrow \mathcal{S} \cap \partial F(M) \right\}.$$

The number j_c is well defined for any $c \in \mathcal{S} \cap \partial F(M)$, and it is $\leqslant n-1$: if $j_c = n$ that means $c \in F(\mathring{M})$ which is impossible.

Next, we will show that $\pi^{j_c+1}(F(M), c)$ is non-trivial. Since $c \in \mathcal{S}$, we know that there exists a neighborhood of c in $\mathbb{R}^n$ such that, for $B_i = F(A_i)$, $\mathcal{V}(A_i)$ a tubular neighborhood of A_i and $V(B_i) = F(\mathcal{V}(A_i))$, we have $F(M) = \cup_{i=1}^{N} V(B_i)$ and $\imath(\mathcal{B}^{j_c}(c, \varepsilon)) = \cap_{i=1}^{N} V(B_i)$.

We can take two $V(B_i)$, that we call V_1 and V_2, to construct a non-trivial element of $\pi^{j_c+1}(F(M), c)$. There exists a tubular neighborhood $\mathcal{U} = \tilde{\imath}(\mathcal{B}^{j_c}(c, \varepsilon) \times] - \varepsilon, \varepsilon[)$ such that $\mathcal{B}^{j_c}(c, \varepsilon)$ splits $\mathcal{U}$: $\mathcal{U}_{+\varepsilon} = \tilde{\imath}(\mathcal{B}^{j_c}(c, \varepsilon) \times \{\varepsilon\}) \subseteq \mathring{V}_1$ and $\mathcal{U}_{-\varepsilon} = \tilde{\imath}(\mathcal{B}^{j_c}(c, \varepsilon) \times \{-\varepsilon\}) \subseteq \mathring{V}_2$. Now, since $F(\mathring{M})$ is contractible, there exists an j_c-homotopy path Γ connecting $\mathcal{U}_{-\varepsilon}$ and $\mathcal{U}_{+\varepsilon}$ in $F(\mathring{M})$. If we glue this path with $\mathcal{U}$: $\Gamma \cup \mathcal{U}$, we have a $j_c + 1$-path, and it cannot be contracted to c by homotopy: indeed, that would imply the existence of a $j_c + 1$-disk in $\mathcal{S}$ near c, thus contradicting the definition of j_c.

Now, the non-triviality of $\pi^{j_c+1}(F(M), c)$ contradicts Corollary 4.1.16. This concludes the proof.

□

With the connectedness of the fibers now proved and remark 4.1.3, we get the following Theorem, which will prove useful in the next chapter as it sheds a new light on the natural structure behind the image of the moment map. Remind first with Theorem 3.1.14 that the sets of critical points of given Williamson type stratifie M by symplectic spaces. We now have

Theorem 4.2.7. *Let (M, ω, F) be an integrable system of dimension 6. The map*

$$\begin{aligned} \mathbf{CrV(F)} : \mathcal{W}_0^n(F) &\to \{\mathrm{CrV}_{\Bbbk}(M, F) \mid \Bbbk \in \mathcal{W}_0(F)\} \\ \Bbbk &\mapsto \mathrm{CrV}_{\Bbbk}(M) \end{aligned}$$

stratifies $F(M)$ by manifolds : $(F(M), \mathbf{CrV(F)})$ is a stradispace for its decomposition by the $CrV_{\Bbbk}(F)$, with the $\Bbbk$'s in $\mathcal{W}_0^n(F)$ (see definition 1.2.19).

Moreover, F is a morphism of stradispaces, in the sense that we have the following diagram

$$\begin{tikzcd} M \arrow[r, "F"'] & F(M) \\ \mathcal{W}_0^n(F) \arrow[u, "\mathbf{CrP(F)}"] \arrow[r, dashed, "id"] & \mathcal{W}_0^n(F) \arrow[u, "\mathbf{CrV(F)}"] \end{tikzcd}$$

and we have that F is a locally trivial fibration on each stratum. More precisely the following restriction/corestriction of $F : \mathrm{CrP}_{\Bbbk}(M) \to \mathrm{CrV}_{\Bbbk}(M)$ gives a torus fibration.

The content of Section 4.1.1 is that the image of the moment map is “not far away” from a polytope.

Chapter 5

Of the integral affine structure of the base space

In this chapter, we describe the base space of the singular Lagrangian foliation associated to an integrable Hamiltonian system by showing it can be endowed with a "singular" integral affine structure. We will use extensively the notion of stradispace introduced in Section 1.3.2.

First we precise our notion of a singular integral affine structure by defining the $\mathbb{Z}$-affine stradispaces. Then we show that on $\mathcal{B}$ we can define naturally such a structure. We can then define a notion of "intrinsic convexity" with respect to this structure, and we show that $\mathcal{B}$ is intrinsically locally convex.

This chapter was initially conceived as a joint work with Nguyen Tien Zung, but we only manage to provide preliminary result to our original goal, which was to give another proof of the conectedness of the fibers using the notion of intrinsic convexity along with Zung's sheaf on the base space. Indeed, our strategy was to use the theorems in [Zun06a] to get a "local-to-global" principle for the convexity of the base space and thus show the **global** intrinsic convexity of $\mathcal{B}$. More generally, this chapter still has non-trivial gaps that need to be filled, but we prefer to see it as questions and conjectures over the subject we worked on. We will indicate them with a **Question** or a **Conjecture** mention.

Anyway, showing intrinsic convexity is of crucial importance to us. First, because compared to the description Theorem we gave in Chapter 4, it provides a more *conceptual* framework, and having a simpler proof of the connectedness of the fibers couldn't be overlooked.

One good reason to maintain the proofs of the Theorems 3.2.10, 4.1.1 and 4.2.4 is that they deal with the **Image of the Moment Map** rather than the base space of the Lagrangian fibration like in [Zun06b] or [Sym01]. The Image of the Moment Map, and its links with the joint spectrum of Complete Set of Commuting Observables is of great importance in Quantum Mechanics. It is the quantum counterpart of the moment map and its image for integrable Hamiltonian systems. It is the fundamental motivation of all the work we have accomplished during this thesis. In particular, all the work in semi-classical analysis accomplished by my Ph.D. advisor as well as other people like Tudor Ratiu or Alvaro Pelayo are still a motivation to produce new theorems like Theorem 4.1.1 that concern the image of the moment map. For instance, there is the joint work of San Vũ Ngọc, Álvaro Pelayo and Leonid Polterovitch in [PPVN13] that shows how the joint spectrum coincides in the semi-classsical limit with the IMM.

We also expect Zung's sheaf and intrinsic convexity to provide a way to recover a family of rationnal convex polytopes from the image of the moment map, the same way San Vũ Ngọc recovered them for the dimension $2n = 4$ in [VN07]. This last fact is yet to be proven, but this is one of the developpements we should work on immediately once this thesis is finished. If we could prove it, this would solve part of the Delzant classification conjecture in the semi-toric case.

5.1 Of singular integral affine spaces

5.1.1 Integral affine structure of a manifold

Let X be a smooth manifold of dimension n. We give three equivalent definitons of an integral affine structure for X.

Definition 5.1.1. *An integral affine structure (or $\mathbb{Z}$-affine structure) on X is given by the data of a lattice $\mathcal{R}_x$ of dimension n on each tangent space depending smoothly of the base point x. Moreover, we ask the family $\mathcal{R} = (\mathcal{R}_x)_{x\in X}$ to be covariant with respect to a torsion-free flat connection ∇ on X.*

Definition 5.1.2. *An integral affine structure is given by a maximal atlas of charts such that the transition functions belong locally to $GL_n(\mathbb{Z}) \rtimes \mathbb{R}^n$.*

The third definition is given in the language of sheaves, and it turns out that this is the formulation that we'll use later on to define singular affine structures.

Definition 5.1.3. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is affine (resp. $\mathbb{Z}$-affine) if it is of the form:*

$$f(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n + b$$

where $(a_1, \dots, a_n) \in \mathbb{R}^n$ (resp. $\mathbb{Z}^n$) and $b \in \mathbb{R}$.

We denote by $\mathrm{Aff}_{\mathbb{R}}(\mathbb{R}^n)$ (resp. $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$), the sheaf of locally ($\mathbb{Z}$ -) affine functions of $\mathbb{R}^n$.

Definition 5.1.4. *An affine structure (resp. $\mathbb{Z}$-affine structure) of dimension n on a Hausdorff topological space X with countable basis, is a subsheaf $\mathcal{A}$ (resp. $\mathcal{R}$) of the sheaf of continuous functions on X, such that the pair $(X, \mathcal{A})$ (resp. $(X, \mathcal{R})$) is locally isomorphic to $(\mathbb{R}^n, \mathrm{Aff}_{\mathbb{R}}(\mathbb{R}^n))$ (resp. $(\mathbb{R}^n, \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n))$).*

Here, when we say that it is locally isomorphic, we may say that it is locally isomorphic *as a differential space*. Again, we use here the language of sheaves to fix a notion of "regular" functions, and then define the spaces which admits functions that "locally looks like" the "regular" functions.

5.1.2 Definition of (integral) affine stradispace

As for the definition of a stradispace, we define the affine, and integral affine (or $\mathbb{Z}$-affine) stradispace by recursion:

Definition 5.1.5. *An (integral) affine structure on a stradispace $(X, \mathbf{C}, \mathcal{S})$ is a vector spaces (ressp. free abelian groups) subsheaf $\mathcal{A}$ (resp. $\mathcal{R}$) of the sheaf $\mathbf{C}$ defined in Definition 1.3.10 such that*

1. *$(S_i, \mathcal{A}(S_i))$ is an (integral) affine manifold of dimension i. Hence we shall write $\mathrm{Aff}_{\mathbb{R}}(S_i)$ (resp. $\mathrm{Aff}_{\mathbb{Z}}(S_i)$) for $\mathcal{A}(S_i)$ (resp. $\mathcal{R}(S_i)$).*
2. *Splitting condition: if $x \in S_i$, for a neighborhood U_x of x there exists a disk $\mathbf{D}^i \subset \mathbb{R}^i$ and a cone over a $(n-i-1)$-dimensional stratified (integral) affine space $\mathbb{A}$ such that, with the cone $\mathcal{C}(\mathbb{A})$ endowed with the (integral) affine structure $\mathcal{A}(\mathcal{C}(\mathbb{A}))$ induced by $\mathcal{A}$ on $\mathcal{C}(\mathbb{A})$ we have:*

$$(U_x, \mathcal{A}(U_x)) \text{ and } \left(\mathbf{D}^i \times \mathcal{C}, \mathrm{Aff}_{\mathbb{R}}(\mathbf{D}^i \to \mathbb{R}) \times \mathcal{A}(\mathcal{C}(\mathbb{A}))\right)$$
$$\left(\text{resp.} (U_x, \mathcal{R}(U_x)) \text{ and } \left(\mathbf{D}^i \times \mathcal{C}, \mathrm{Aff}_{\mathbb{Z}}(\mathbf{D}^i \to \mathbb{R}) \times \mathcal{R}(\mathcal{C}(\mathbb{A}))\right)\right)$$

are isomorphic as differential spaces.

3. *Point differentiation: For each $x, y \in X$ there is a section $\alpha \in \mathcal{A}$ (resp. $\alpha \in \mathcal{R}$) such that $\alpha(x) = 0$ and $\alpha(y) \neq 0$.*

It is the triplet $(X, \mathcal{A}, \mathcal{S})$ (resp. $(X, \mathcal{R}, \mathcal{S})$) that is called an (integral) affine stradispace.

As for the definition of a symplectic stradispace, for a ($\mathbb{Z}$-)affine stradispace the most important is that the strata are ($\mathbb{Z}$-)affine manifolds, and the splitting condition.

5.1.3 The base space is an integral affine stradispace

A consequence of 3.1.11 is that $\mathcal{B}$ equipped with the quotient topology is a Hausdorff space. It is locally compact since M is compact.

Topological and affine monodromy representation by suspension

First, let's give a definition of topological monodromy which is due to Symington (see [LS10], and [Sym01]).et $\mathcal{M}(\mathbb{T}^n)$ denote the mapping class group of the n-torus, that is, the set of isotopy classes of self-diffeomorphisms of the torus. We have that $\pi_{\mathcal{F}} : M^{2n} \to \mathcal{B}_r$ is a $\mathbb{T}^n$-bundle. A monodromy representation of $\pi_{\mathcal{F}}$ is a homomorphism $\psi : \pi_1(\mathcal{B}_r, b) \to \mathcal{M}(\mathbb{T}^n)$ such that for any $[\gamma] \in \pi_1(B_r, b)$ represented by a based loop $\gamma : S^1 \to B_r$, the total space of $\gamma^* \pi_{\mathcal{F}}$ is diffeomorphic to the suspension of $\psi[\gamma]$:

$$I \times \mathbb{T}^n \diagup_{(0,q) \sim (1, \psi[\gamma](q))}.$$

We can then identify each isotopy class of $\psi[\gamma]$ with the induced automorphism on $H^1(\mathbb{T}^n, \mathbb{Z})$. This automorphism is uniquely defined. Choosing a basis for $H^1(\mathbb{T}^n, \mathbb{Z})$ we then get an induced morphism group from $\pi_1(B, b)$ into $GL_n(\mathbb{Z})$. Of course, the two definition of the monodromy representation coincide.

Strictly speaking, we need to fix a base point b_0 to speak of the fundamental group $\pi_1(\mathcal{B}, b_0)$. Otherwise it is the fundamental groupoid π_1. However, in our almost-toric case, $\mathcal{B}_r$ is connected so all fundamental groups $\pi_1(\mathcal{B}_r, b_0)$ are isomorphic and we can thus speak of "the" fundamental group. From now on, by the topological, or homological monodromy of a regular torus fibration we mean the induced map from $\pi_1(\mathcal{B}_r)$ to $GL_n(\mathbb{Z})$, again by taking the cohomology class of the integration over the n fundamental circles as a basis for $H^1_{dR}(\mathbb{T}^n, \mathbb{Z})$.

Since $\mathbb{T}^n$ has a canonical flat connection, the vector bundle $E_{\mathbb{R}} \xrightarrow{H^1_{dR}(\mathbb{T}^n,\mathbb{R})} \mathcal{B}_r$ can be indentified with the bundle of constant vector fields on the fibers of $M_r \xrightarrow{\mathbb{T}^n} \mathcal{B}_r$: on each fiber we push the vector X from the origin by the affine connection to all the points of the fiber and get a vector $\tilde{X}_c$ on each point of $\mathbb{T}^n_c$. Next, to a $\mathbb{T}^n$-constant vector field $\tilde{X}$, we can associate the 1-form $\alpha(\tilde{X}) = \omega(\tilde{X}, .)$. The map

$$E_{\mathbb{R}} \to T^*\mathcal{B}_r$$
$$X \mapsto \alpha(\tilde{X})$$

is an isomorphism. So the Gauss-Manin connection defined previously can be defined on $T^*\mathcal{B}_r$. On the other hand, it is easy to see that Liouville-Arnold-Mineur Theorem 1.2.14 provides an atlas of charts with transition functions in $GA_n(\mathbb{Z})$, that is, an (integral) affine structure. We can take the dual of the affine connection it defines on the tangent bundle, and see that the two coincide.

The discrete subbundle $E_{\mathbb{Z}} \to \mathcal{B}_r$ of $E_{\mathbb{R}} \to \mathcal{B}_r$ can be identified with our isomorphism above to a "discrete" subbundle of $T^*\mathcal{B}_r$, consisting of "integral" covectors. This is what we call the period lattice, and this is the essence of integrable systems : wherever they are toric, periods are both integral affine vectors and 1-forms.

Duistermaat sheaf VS Zung sheaf

Several sheaves were defined over the years on the base space of the Lagrangian foliation by authors trying to investigate the singular Lagrangian fibration. The motivation was to determine the obstruction of piecing together the local Action-Angle coordinates given by Theorem 1.2.14 around regular values to get global Action-Angle coordinates. To answer this question they used **obstruction theory**, whose philosophy is to define geometrical objects that encapsulate the different kind of obstacles existing to global Action-Angle coordinates. This is another approach than the one we illustrated in Section 3.3, where we constructed explicit Action-Angle coordinates with singularity for two of them at critical points with focus-focus components. The ambiguity in the choice of the action-angle coordinates in Section 3.3 was exactly the manifestation of monodromy, which is one obstacle to the existence of global Action-Angle coordinates, but not the only one. We shall follow here the path of [Dui80b] and [Zun03], and define here monodromy by obstruction theory.

The first sheaf to be defined that dealt with global Action-Angle coordinates was defined by Duistermaat in [Dui80b], and it was defined only on the regular part of the Lagrangian fibration.

For a general singular locally trivial fibration $\pi : M \to O$ with typical (we may say regular) fiber F, if we write S the set of singular base points, then $\pi : M \backslash \pi^{-1}(S) \to O \backslash S$ is a regular locally trivial fibration. If we now exchange each regular fiber $P_c := \pi^{-1}(c)$ by its first homology group over $\mathbb{Z}$, we get a fiber bundle which is integral affine. We have a vector bundle $h_\pi : E_{\mathbb{K}} \to O \backslash S$. Since the fiber of the initial fibration are homotopical to each other, we can move the cohomology classes of one fiber of h_π to cohomology classes of nearby fibers. This gives us a notion of (locally) flat section for the bundle h_π. The induced locally flat connection induced on this bundle is called the Gauss-Manin connection. The holonomy of this connection is a group morphism from the fundamental group $\pi_1(O \setminus S)$ to $\mathrm{Aut}(H^*_{dR}(P_c, \mathbb{K}))$, and this is what we define as the topological monodromy of the fibration π.

An integrable Hamiltonian system with its singular Lagrangian fibration $\pi_{\mathcal{F}} : M^{2n} \to \mathcal{B}$ is just a particular case of the general construction above, where $P_c \simeq \mathbb{T}^n$ for all $c \in \mathcal{B}_r := \mathrm{CrL}_{X^n}(\mathcal{B})$ the set of regular base points (regular leaves). In this case we take, following Duistermaat, only the first cohomology group with integer coefficients $H^1_{dR}(P_c, \mathbb{Z})$. We get an integral lattice bundle $E_{\mathbb{Z}} \to_{H^1_{dR}(P_c,\mathbb{Z})} \mathcal{B}_r$. Then we fix a base point, otherwise monodromy will only be defined up to conjugacy.

The sheaf of local sections of $E_{\mathbb{Z}} \to_{H^1_{dR}(\mathbb{T}^n,\mathbb{Z})} \mathcal{B}_r$ is called Duistermaat linear monodromy sheaf. It is a locally constant sheaf, and it is completely determined by the monodromy representation ψ.

We then define, now following [Zun03], another sheaf that is now defined also on the critical points.

Definition 5.1.6 ([Zun03])**.** *Let F be a non degenerate integrable system, $\pi_{\mathcal{F}} : M^{2n} \to \mathcal{B}$ its singular Lagrangian foliation. We define the sheaf $\mathcal{R}$ by:*

$$\mathcal{R}(U) := \left\{ \begin{array}{c} \rho : S^1 \times \pi^{-1}(U) \to \pi^{-1}(U) \text{ a Hamiltonian action of } S^1 \text{ s.t.} \\ \forall \theta \in S^1, \forall m \in \pi^{-1}(U), F(\theta \cdot_\rho m) = F(m) \end{array} \right\}.$$

$\mathcal{R}(U)$ is a free Abelian group.

We have the theorem :

Theorem 5.1.7. *Let $(M^{2n}, \omega, F$ be a non-degenerate semi-toric integrable system, $\pi_{\mathcal{F}} : M^{2n} \to \mathcal{B}$ its singular Lagrangian foliation,*

$$\mathcal{S} = \mathbf{CrL(F)} : \begin{array}{cc} \mathcal{W}_0^n(F) \to & \left\{ \mathrm{CrL}_{\Bbbk}(M) \mid \Bbbk \in \mathcal{W}_0^n(F) \right\} \\ \Bbbk \mapsto & \mathrm{CrL}_{\Bbbk}^F(M) \end{array}$$

its stratification by the Williamson type of the leaf. We have that $(\mathcal{B}, \mathcal{R}, \mathcal{S})$ is a $\mathbb{Z}$-affine stradispace.

Proof. With Theorem 3.1.14, we already have that the Williamson type $\Bbbk$ of critical points stratifies M by symplectic strata of dimension $2k_x$. When we consider the restriction of F to the strata $A_{\Bbbk}^{(i)}$ that we introduced in Theorem 4.1.1, $F_{\Bbbk}^{(i)} = F \circ A_{\Bbbk}^{(i)}$, we get that for $\mathcal{U}$ an open neighborhood in $A_{\Bbbk}^{(i)}(M)$, $F_{\Bbbk}^{(i)}\big|_{\mathcal{U}}$ yields a Hamiltonian $\mathbb{T}^{n-1}$-action of rank $\check{k}_x^{(i)}$. The points of $A_{\Bbbk}^{(i)}(M)$ are regular points for this Hamiltonian action. The components of $F_{\Bbbk}^{(i)}$ are a basis of $\mathcal{R}(\pi_{\mathcal{F}}(\mathcal{U}))$, and by Arnold-Liouville, we have thus that $\mathcal{R}(\pi_{\mathcal{F}}(\mathcal{U}))$ is isomorphic to $\mathrm{Aff}_{\mathbb{Z}}(V)$ where V is an open neighborhood of $\mathbb{R}^{\check{k}_x^{(i)}}$.

To prove the splitting condition for the base space, we rely on the proof of the splitting condition we gave for $(M, \mathcal{C}^\infty(M \to \mathbb{R}), \mathbf{CrP(F)})$, but here in the special case where $k_h = 0$. Again, with Item 2. of Theorem 3.1.11 we see that we only need to treat the simple elliptic and focus-focus cases, but this time with the models of the leaves.

Let's consider a $b \in \mathcal{B}$ a base point. In the $n = 1$ elliptic case, $\mathrm{CrL}_E(V_b)$ is a point. A neighborhood of it is homeomorphic in $\mathcal{B}$ to a disk of dimension 0: again, a point. Let's treat also a less trivial example: in the $n = 2$, for b being the $E - E$ case, $\mathrm{CrL}_{E-E}(V_b)$ is, with no change, homeomorphic in $\mathcal{B}$ to a point, but this time, a neighborhood of $\mathrm{CrL}_{E-E}(V_b)$ is homeomorphic to a half-open segment times a half-open segment $[0, 1[\times [0, 1[$. This can be seen as the cone over a closed segment :

$$[0,1] \times [0,1[\Big/ [0,1] \times \{0\}.$$

So the splitting condition holds also for $E - E$ critical leaves.

In the $n = 2$ focus-focus case, $\mathrm{CrL}_{FF}(\mathbb{R}^4)$ is a pinched torus, that is, a sphere with two endpoints identified. Its image by $\pi_{\mathcal{F}}$ is a point. A "singular" tubular neighborhood of the pinched torus is homeomorphic to a disk in $\mathcal{B}$, with here $n = 2$, and $i = 0$. Thus the disk can be seen as a cone over the $n - i - 1$-dimensional stradispace of regular points, and it is endowed with an integral affine structure, which is non-trivial when we turn around focus-focus points.

Last thing to mention is that, if we are on the base point of a leaf with transverse components X^r, we can extend the integral affine structure along these components. This proves that there exists a disk $\mathbf{D}^r \subset \mathbb{R}^r$ and a cone over a $n-r-1$-dimensional stratified (integral) affine space $\mathbb{A}$ such that, with the cone $\mathcal{C}(\mathbb{A})$ endowed with its integral affine structure $\mathcal{R}(\mathcal{C}(\mathbb{A}))$ induced by $\mathcal{R}$ on $\mathcal{C}(\mathbb{A})$ we have that $(U_x, \mathcal{R}(U_x))$ and $(\mathbf{D}^r \times \mathcal{C}, \mathrm{Aff}_{\mathbb{Z}}(\mathbf{D}^r \to \mathbb{R}) \times \mathcal{R}(\mathcal{C}(\mathbb{A})))$ are isomorphic as differential spaces : they are *homeomorphic*. This concludes the proof of our theorem.

$\square$

Theorem 5.1.7 means that the base space of a semi-toric integrable Hamiltonian system is an integral affine stradispace. Zung's sheaf can also be designated by the letter **A**: the **A** would stand here for "actions", since the sheaf here coincides with the sheaf of closed action 1-forms.

5.2 Local and global intrinsic convexity

5.2.1 Affine geodesics and convexity

Given an affine structure, one can define affine geodesics on X in two steps as following:

Definition 5.2.1. *We define the* **regular** *affine geodesics between two points x and y in X, as 1-dimensional compact differential embedded subspaces γ with endpoints x and y and such that there exists $n-1$ functions*

$$u_1, \ldots, u_{n-1} \in \mathcal{A}$$

that are linearly independent everywhere and that are constant on γ.

Definition 5.2.2. *We define the affine geodesics between two points x and y in X as the 1-dimensional compact differential spaces γ with endpoints x and y that are uniform limits of regular affine geodesics.*

We'll see that even though **regular** affine geodesics are simply 1-dimensional subspaces linking x and y, when the affine structure is non-trivial, for instance, when it has non-trivial monodromy, there can be affine geodesics that are non-trivial. As a result, the "straight lines" that are geodesics can be broken lines at the singular points. The resulting geometry can still be convex, but there are examples where geodesics are non-convex.

Example 5.2.3 (Non-convex geometry)**.** *Let's assume that, for this $\mathbb{Z}$-affine stradispace, the affine monodromy is given by the matrix*

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

A local model where this situation occurs is when instead of cutting out an angular sector and glue together the two edges as in the focus-focus case, we remove a half-line starting at one point and add a sector, as it is explained in the drawing below.

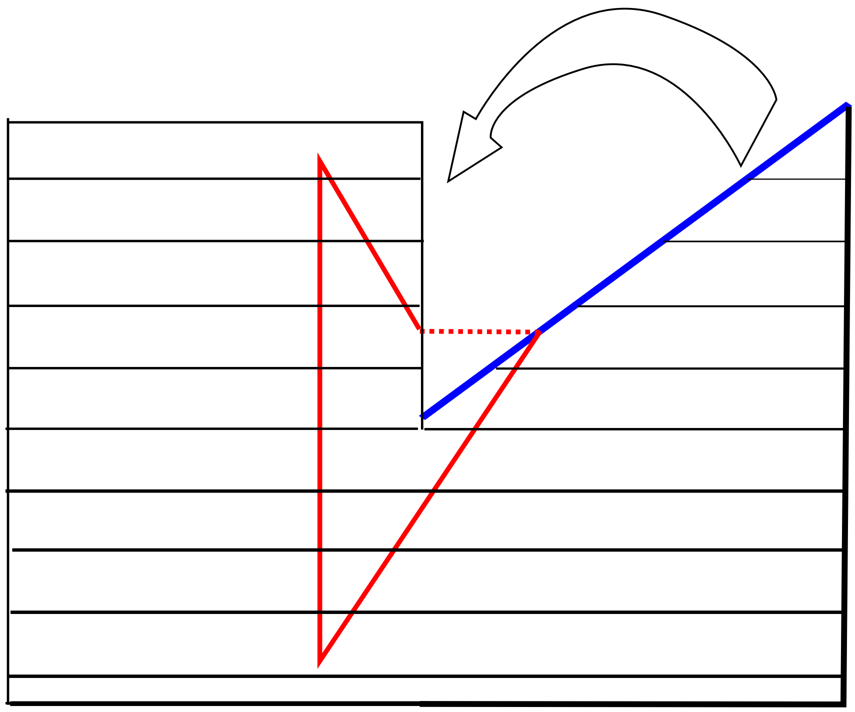

Figure 5.1: An example of convex geometry

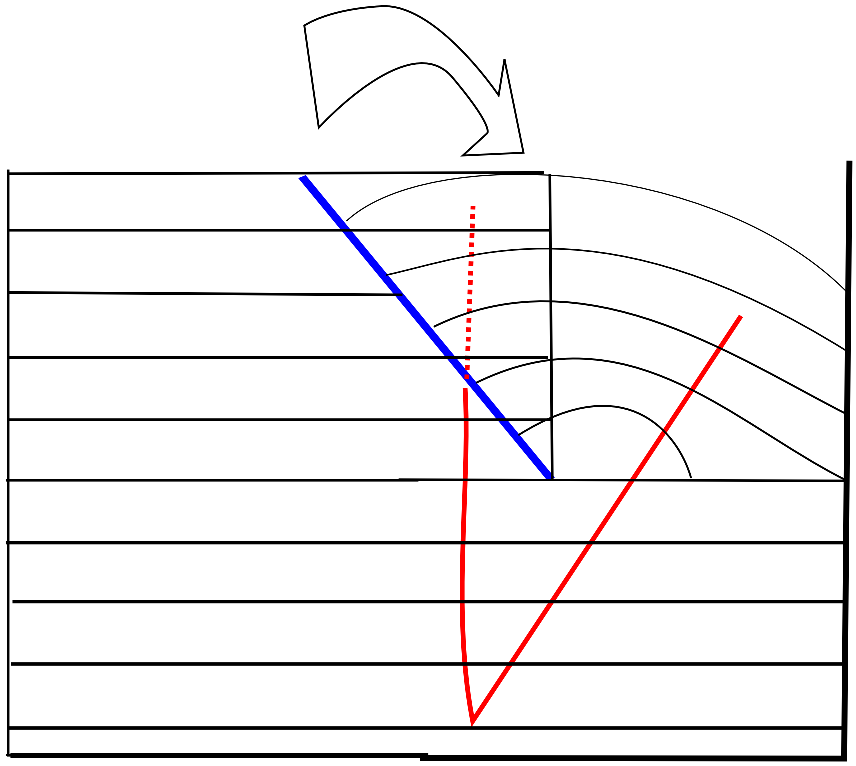

Figure 5.2: An example of non-convex geometry

The affine monodromy makes it impossible for some point to be connected by affine geodesics.

Question 5.2.4. *1. Does all regular geodesics lie in $S_n = \mathrm{CrL}_{X^n}(M)$?*

2. Can we say that the regular geodesics are the ones that can be well defined using the notion of a connection ∇ on a differential space ? If a geodesic lies entirely in $A^{(i)}_{\Bbbk}$, does it mean that it is a geodesic in the "usual sense" for $\nabla^{(i)} = \nabla|_{A^{(i)}_{\Bbbk}}$?

Definition 5.2.5. *An affine stradispace $\mathcal{B}$ is called* **intrinsically convex** *if for any two points p_1, p_2 in $\mathcal{B}$ there exists an affine geodesic curve on $\mathcal{B}$ which goes from p_1 to p_2. $\mathcal{B}$ is called* **locally convex** *if every point of $\mathcal{B}$ admits a generating system of neighborhoods which are intrinsically convex.*

For instance, with an affine monodromy $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, on has a convex geometry, even if there can be more than one geodesic between two points.

5.2.2 The base space of an almost-toric IHS is intrinsically convex

Theorem 5.2.6. *The base space of a nondegenerate IHS of almost-toric type on a compact symplectic manifold is intrinsically convex and contractible.*

Remark 5.2.7. *When there are no focus-focus singularities, then the above theorem is a well-known result (Delzant, Molino, Boucetta) and can be viewed as a particular case of the Atiyah-Guillemin-Sternberg convexity theorem (modulo the proof of the fact that there is a global $\mathbb{T}^n$-action).*

In the case with 2 degrees of freedom and with focus-focus singularities, this theorem was observed by Zung in his thesis (see also paper [Zun03]).

As we said, our strategy for proving this theorem was based on the local convexity of the base space, and the "local-to-global" principle. We will just show the local convexity of the base space.

Proposition 5.2.8. *Let $(M^{2n}, \omega) \xrightarrow{\pi} (B, \mathcal{R}, \mathcal{S})$ be an almost-toric integrable system. The base space is locally convex as an affine stradispace.*

Proof. First, it is easy to see that it suffices to look at the singularities of the almost-toric system to prove the result. Since we only deal with fibers that are product of simple, elementary and topologically stable singularities, we can use Theorem 3.1.11, and bring back the original proposition to proving that the local models for the affine structure of the base space is intrinsically convex with respect to the affine structure induced by the integrable system. We first prove the following lemma

Lemma 5.2.9. *The product of two intrinsically convex affine stradispaces, endowed with the canonical affine structure, is again an intrinsically convex affine stradispace.*

Proof. It is enough to prove it for points linked by regular geodesics, and then take the limit to treat the remaining points. First, we prove it for the product of affine stradispaces $\mathcal{B} \times \mathcal{B}'$, endowed with the direct product of the affine structures of $\mathcal{B}$ and $\mathcal{B}'$.

Let $(B, \mathcal{A}, \mathcal{S})$, $(B', \mathcal{A}', \mathcal{S}')$ be two intrinsically convex affine stradispaces, $(x, x'), (y, y') \in B \times B'$. There exists a one-dimensional differential space γ and sections $u_1, \ldots, u_{n-1} \in \mathcal{A}$ that are linearly independent everywhere on B and that are constant on γ.

We can choose the u_i's such that $\gamma \subseteq \bigcup_{i=1}^{n-1} \ker(u_i) = D$. We complete the free family with u_n that is linearly independent from the others (there is a one-dimensional family of such u_n). We can set the values of u_n on x and y : $u_n(x) = 0$ and $u_n(y) = 1$.

We do the same construction for B' and get a basis $u'_1, \ldots, u'_{n'}$, and a one-dimensional differential embedded sub-stradispace γ' with

$$\gamma' \subseteq \bigcup_{i=1}^{n-1} \ker(u'_i) = D'.$$

Now, on $B \times B'$, the canonical affine structure is given by:

$$f \in \mathcal{A} \times \mathcal{A}' \text{ if } f = a \circ \pi_B + a' \circ \pi_{B'} \text{ with } a \in \mathcal{A} \text{ and } a' \in \mathcal{A}'.$$

The basis $(u_1, \dots, u_n)$ and $(u'_1, \dots, u'_{n'})$ inject naturally in $\mathcal{A} \times \mathcal{A}'$. As a result, we have $n + n' - 2$ functions $(\tilde{u}_i = u_i \circ \pi_B)_{i=1..n-1}$ and $(\tilde{u'}_i = u'_i \circ \pi_{B'})_{i=1..n'-1}$ that are constant on $\gamma \times \gamma' \subseteq D \times D'$, which is a two-dimensional differential embedded sub-stradispace. We need to find one extra function that is linearly independent from the $\tilde{u}_i$'s and $\tilde{u'}_i$'s and that is constant on a one-dimensional differential embedded sub-stradispace containing (x, x') and (y, y'), that is to be determined.

The function $\tilde{u}_n - \tilde{u'}_{n'}$ is clearly linearly independent of the functions $(\tilde{u}_i, \tilde{u'}_i)_{i=1..n-1}$: it is in $Vect(\tilde{u}_n, \tilde{u'}_n)$ and it is non-zero. Hence, its kernel intersects transversally $\gamma \times \gamma'$. The intersection is a one-dimensional differential embedded sub-stradispace which contains (x, x') and (y, y'):

$$(\tilde{u}_n - \tilde{u'}_{n'})(x, x') = 0 - 0 = 0\ ;\ (\tilde{u}_n - \tilde{u'}_{n'})(y, y') = 1 - 1 = 0.$$

□

However, the affine structure is not, in general, a direct product of affine structures. Hence, we must prove a parametrized version of Lemma 5.2.9, that is:

Conjecture 5.2.10. *Let $(B, \mathcal{A}, \mathcal{S})$ be an intrinsically convex affine stradispace, and $(B', \mathcal{A}', \mathcal{S}')$ another instrinsically affine stradispace such that $B' = H(B)$ with H a local affine isomorphism. Then the product of the two stradispaces $B \times B'$ is again an intrinsically convex affine stradispace.*

Nguyen Tien Zung and I hope to come up with a proof of this conjecture very soon. We believe it should follow the path of the direct product case, but instead of taking the linear function $u - u'$, take $u - H(u)$.

Once proved, the project of Nguyen Tien Zung and I goes on the path of intrinsic convexity. We'd show the following conjecture:

Conjecture 5.2.11. *The local models of the base space near regular, elliptic, and focus-focus values are intrinsically convex.*

The affine local models should be a consequence of Theorem 3.1.11:

For a regular value p: we have with the classical Arnol'd-Liouville that there exists a neighborhood of p endowed with its affine structure $\mathcal{A}$ which is isomorphic to $(]p - \epsilon, p + \epsilon[, \mathrm{Aff}_{\mathbb{R}}(\mathbb{R}^1))$, obviously convex.

For an elliptic value p: we have with Miranda-Zung that there exists a neighborhood of p endowed with its affine structure $\mathcal{A}$ given by the action of T^*B on M which is isomorphic to $([p, p + \epsilon[, \mathrm{Aff}_{\mathbb{R}}(\mathbb{R}^1))$, again convex.

For a focus-focus value p: this is the hardest part. Here, we have a two-dimensional model that does not immerse affinely in $(\mathbb{R}^2, \mathrm{Aff}_{\mathbb{R}}(\mathbb{R}^2))$. This non-triviality of the affine structure is due to the presence of non-trivial monodromy. We currently don't have established a proof of the local convexity in the focus-focus case for the present time, we know that Nguyen Tien Zung has checked it but it still need to be written down carefully.

Lemma 5.2.12. *For all $u \in \mathbb{R}^2$, $(\mathcal{V}_u, \mathcal{A}_u)$ is affine isomorphic to $(\mathcal{V}_{\vec{e}_1}, \mathcal{A}_{\vec{e}_1})$.*

Proof. It can be proved by a simple change of basis. □

Lastly, to prove the global intrinsic convexity of the base space, we conjecture we can use arguments similar to the proof of Lemma 3.7 of [Zun06a].

□

5.3 The semi-toric case

Here we explain what we expect showing the intrinsic convexity of the base space, and a quick sketch of the proof we intend to use. It is actually no less than the Conjecture 4.1.5, here proven with a new set of techniques.

Conjecture 5.3.1. Connectedness of the fibers for semi-toric systems of dimension $2n$ *Let* $\mathbf{F} = (H = F_1, F_2, \ldots, F_n) : (M^{2n}, \omega) \to \mathbb{R}^n$ *be the momentum map of a nondegenerate IHS of semitoric type on a compact symplectic manifold* (M^{2n}, ω)*, such that* $(F_2, \ldots, F_n) : (M^{2n}, \omega) \to \mathbb{R}^{n-1}$ *generates a global Hamiltonian* $\mathbb{T}^{n-1}$*-action on* (M^{2n}, ω)*.*

Then the image $\mathbf{F}(M^{2n})$ *of the momentum map* $\mathbf{F}$ *is a contractible closed domain of* $\mathbb{R}^n$*, and the preimage of each point by* $\mathbf{F}$ *is a connected subset of* M^{2n}*.*

Sketch of the proof for the Conjecture 5.3.1 Denote by $\mathcal{B}$ the base space. Then $\Phi = (\hat{F}_2, \ldots, \hat{F}_n) : \mathcal{B} \to \mathbb{R}^{n-1}$ is an affine map. We want to show that $\hat{\mathbf{F}} : \mathcal{B} \to \mathbf{F}(M^{2n})$ is injective.

If we assume the contrary, then there are two different points $z_1 \neq z_2$ in $\mathcal{B}$ which have the same image with respect to the momentum map. Denote by ℓ a maximal affine geodesic curve in $\mathcal{B}$ which passes by z_1 and z_2. Then the image of ℓ via Φ is just one point due to the fact that Φ is affine on ℓ and $\Phi(z_1) = \Phi(z_2)$. Due to the nondegeneracy of the integrable Hamiltonian system, the function H must be monotonous on ℓ, which leads to a contradiction.

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Systèmes intégrables semi-toriques et polytopes moment

Un système intégrable semi-torique sur une variété symplectique de dimension $2n$ est un système intégrable dont le flot de $n-1$ composantes de l'application moment est 2π-périodique. On obtient donc une action hamiltonienne du tore $\mathbb{T}^{n-1}$. En outre, on demande que tous les points critiques du système soient non-dégénérés et sans composante hyperbolique. En dimension 4, San Vũ Ngọc et Álvaro Pelayo ont étendu à ces systèmes semi-toriques les résultats célèbres d'Atiyah, Guillemin, Sternberg et Delzant concernant la classification des systèmes toriques.

Dans cette thèse nous proposons une extension de ces résultats en dimension quelconque, à commencer par la dimension 6. Les techniques utilisées relèvent de l'analyse comme de la géométrie symplectique, ainsi que de la théorie de Morse dans des espaces différentiels stratifiés. Nous donnons d'abord une description de l'image de l'application moment d'un point de vue local, en étudiant les asymptotiques des coordonnées action-angle au voisinage d'une singularité foyer-foyer, avec le phénomène de *monodromie* du feuilletage qui en résulte.

Nous passons ensuite à une description plus globale dans la veine des polytopes d'Atiyah, Guillemin et Sternberg. Ces résultats sont basés sur une étude systématique de la stratification donnée par les fibres de l'application moment. Avec ces résultats, nous établissons la connexité des fibres des systèmes intégrables semi-toriques de dimension 6 et indiquons comment nous comptons démontrer ce résultat en dimension quelconque.

Semi-toric integrable systems and moment polytopes

A semi-toric integrable system on a symplectic manifold of dimension $2n$ is an integrable system for which the flow of $n-1$ components of the moment map is 2π-periodic. We thus obtain a Hamiltonian action of the torus $\mathbb{T}^{n-1}$. Moreover, one asks that all critical points of the system be non-degenerate and without hyperbolic component. In dimension 4, San Vũ Ngọc and Álvaro Pelayo extended to these semi-toric systems the famous results of Atiyah, Guillemin, Sternberg and Delzant concerning the classification of toric systems.

In this thesis we propose an extension of these results to any dimension, starting with the dimension 6. We rely on techniques coming from analysis as well as symplectic geometry, and also Morse theory in stratified differential spaces. We first give a local description of the image of the moment map, by studying the asymptotics of the action-angle coordinates near a focus-focus singularity and the resulting phenomenon of *monodromy* of the foliation.

Then we move towards a more global description, in the spirit of Atiyah, Guillemin and Sternberg polytopes. This description is based on a systematic study of the stratification given by the fibers of the moment map. With these results, we establish the connexity of the fibers for semi-toric integrable systems of dimension 6, and indicate how we expect to prove this result in any dimension.